2
GEODETIC CONCEPTS

This chapter assembles a number of geodetic concepts and formulae that are required in the discussions of the geodetic methods and geophysical problems encountered in later chapters. The discussion of these concepts and derivations of the relevant formulae cannot be complete here, and the reader will have to consult other geodetic references such as Bomford (1971), Heiskanen and Moritz (1967), Levallois (1970), and Vaníček and Krakiwsky (1982). The two principal subjects that are summarized here are (i) aspects of potential theory that are required for a discussion of the shape of the Earth and for the interpretation of the gravity field, and (ii) aspects of reference systems that are required for discussions of the planet's rotation and surface deformations. In addition, this chapter collects together a number of other formulae that are required in later sections.

The theory of the Earth's shape is summarily discussed in section 2.2. The particular equipotential surface chosen to represent the shape of the Earth is the *geoid,* a surface that, at sea, corresponds to within about 1–2 m to the time-averaged ocean surface. In the first approximation this surface can be represented by an ellipsoid of flattening $f=(R_e-R_p)/R_e \simeq 1/300$, where R_e and R_p are the mean equatorial and polar radii respectively of the Earth, and such a surface provides a convenient reference shape for the planet. The departures $\mathcal{N}$ of the geoid from this reference ellipsoid is termed the *geoid height.* For most geodetic work the parameters f and R_e are chosen so as to minimize these geoid departures and $\mathcal{N}$ is then of the order of 100 m. The precision of observations of $\mathcal{N}$, $\sigma_{\mathcal{N}}$, approaches 20 cm and $\sigma_{\mathcal{N}}/R_e \simeq 3\times10^{-8}$. The principal observed quantity that defines the geoid is gravity, g, and any theory of the Earth's reference model must also include a formulation for the gravity field of this reference ellipsoid. For geophysical interpretations the appropriate reference figure, more useful than this best-fitting ellipsoid, is the hydrostatic equilibrium configuration of a rotating body whose size, radial density distribution, and rotational velocity correspond to the actual Earth. A brief discussion of this theory is also required.

To monitor motions of the Earth's rotation axis, or to measure movements and strains of the crust, it is necessary to establish coordinate reference frames with respect to which these movements can be defined and measured. For studies of the Earth's rotation, or for measurements of the global movements of the crust, this reference system is of necessity a global one. For studying crustal deformations of limited area more localized reference frames will suffice, but sooner or later it becomes desirable to relate the local movements into a larger framework of tectonic motions and a global framework will be required. The geodetic reference systems have traditionally been established by a combination of trigonometric surveys with astronomical observations of latitude and longitude and sometimes with gravity observations, but the low accuracy of these measurements (Chapter 5) makes such surveys of little value in monitoring the global deformations. Only through the application of space technology to geodesy, through the use of laser and electronic observations of artificial satellites and the Moon, and by measuring emissions from stellar sources at radio-frequencies, has it become possible to establish the requisite high-precision global reference frames.

The reference frames are of two basic types; a terrestrial frame tied in some way to the Earth, and an astronomically defined frame representing a stationary or inertial coordinate system. Sometimes these systems are established by introducing intermediate reference frames, such as one in which motion about the Earth of artificial satellites are defined. A complete discussion of the reference frames therefore requires discussion of terrestrial, astronomical, and orbital reference systems and also of their relationship through time; of the time-dependence of their orientations arising from the motions of the Earth's rotation axis in space (the precession and nutation) as well as relative to the Earth's crust (the polar motion). Some basic elements of this motion are discussed in sections 2.4 and 2.5 but a more complete geophysical discussion is deferred to Chapter 11. Aspects of these reference frames are discussed in section 2.3. More detailed accounts can be found in the volumes edited by Gaposchkin and Kolaczek (1981), Moritz and Mueller (1986), Babcock and Wilkins (1988), and Kovalevsky *et al.* (1988). The subject is an active one.

2.1. Gravity and gravitational potential

The dominant force that shapes the Earth is gravity; the gravitational attraction from the mass distribution within the planet and the centrifugal force that results from the planetary rotation. In a stationary reference frame X_i $(i = 1, 2, 3)$ the magnitude of the force of attraction $\delta \boldsymbol{F}$ experienced by a unit mass located at $P(X_i)$ caused by an element of mass $d\mathcal{M}'$ at $P'(X_i')$ is given by the universal law of gravitational attraction

$$\delta \boldsymbol{F} = G\boldsymbol{L}L^{-3}\,d\mathcal{M}', \qquad (2.1.1a)$$

where G is the gravitational constant and $\boldsymbol{L}$ is the vector from P′ to P, or

$$\boldsymbol{L} = \sum_i (X_i' - X_i)\hat{\boldsymbol{k}}_i \qquad (2.1.2a)$$

and

$$L = \left[\sum_i (X'_i - X_i)^2\right]^{1/2}, \tag{2.1.2b}$$

where the $\hat{\boldsymbol{k}}_i$ are unit vectors in the directions of the X_i axes. The direction of the force on the unit mass has the direction PP′. For an attracting body of finite dimensions

$$\boldsymbol{F} = G\int_{\mathcal{M}} L^{-3}\boldsymbol{L}\,\mathrm{d}\mathcal{M}', \tag{2.1.1b}$$

where the integral is over the body of mass $\mathcal{M}$. The components F_i ($i = 1, 2, 3$) of the force, parallel to the X_i axes, are

$$F_i = G\int_{\mathcal{M}} L^{-3}(X'_i - X_i)\,\mathrm{d}\mathcal{M}'. \tag{2.1.3}$$

$\boldsymbol{F}$ is the gravitational force. The potential V of the force is defined by

$$\boldsymbol{F} = \operatorname{grad} V = \nabla V \tag{2.1.4}$$

where $\nabla = \Sigma_i(\partial/\partial x_i)$ is the gradient operator. The derivatives of this potential in any direction represent the force components in this direction, and the force field is fully determined by the potential. This sign convention is one widely adopted in astronomy, geodesy, and geophysics (e.g. Kaula 1966*a*, 1968; Brouwer and Clemence 1961; Heiskanen and Moritz 1967; see also Sternberg and Smith 1964) although the sign convention more usual to physics texts is also encountered in the geophysics literature (e.g. Stacey 1977*a*). With (2.1.2) and (2.1.3)

$$V = G\int_{\mathcal{M}} L^{-1}\,\mathrm{d}\mathcal{M}'. \tag{2.1.5}$$

V is the potential of gravitation per unit mass of the attracted body located at $P(X_i)$. For a body of mass $\mathcal{M}$ at $P(X_i)$ the total gravitational potential, or force function, is $\mathcal{M}V$. The potential energy is defined as $-\mathcal{M}V$.

In discussing the gravitational potential of a rotating planet it is convenient to adopt a set of axes x_i that are fixed to and rotate with the body, particularly when the gravitational force is measured at any stationary point $P(x_i)$ on the surface of the rotating body. The x_i system is discussed further in section 2.4. The force $\boldsymbol{g}$ acting at this point is then the vector sum of the attraction $\boldsymbol{F}$ and the centrifugal force, or

$$\boldsymbol{g} = G\int_{\mathcal{M}} L^{-3}\boldsymbol{L}\,\mathrm{d}\mathcal{M} + \omega^2 p\hat{\boldsymbol{p}} \tag{2.1.6}$$

where $\hat{\boldsymbol{p}}$ is the unit vector through P perpendicular to the rotation axis, p

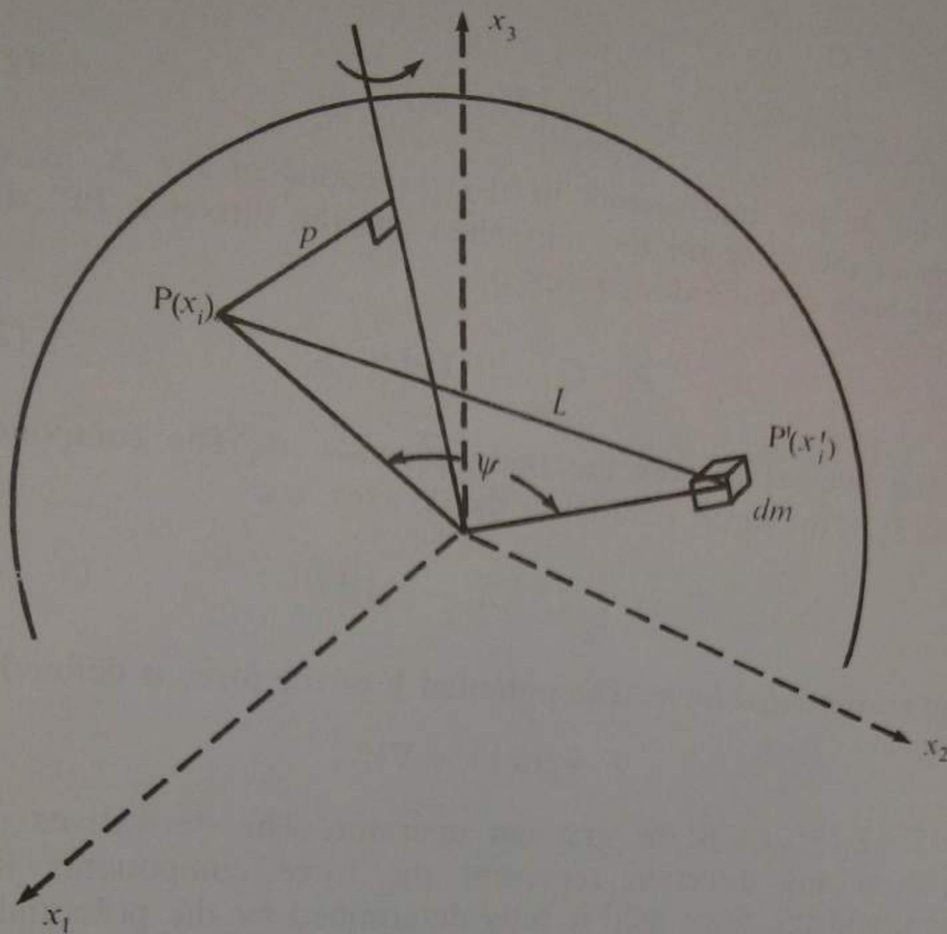

FIG. 2.1. Geometry of the relative positions of an element of mass dm, at P$'(x_i')$, of the Earth and the position P(x_i), at a distance L from P$'$, of a unit mass that is attracted by d$\mathcal{M}$. p is the distance of P from the instantaneous rotation axis $\boldsymbol{\omega}$.

is the corresponding distance, and ω is the rate of rotation about this axis (see Fig. 2.1). This axis is, for convenience, taken to be parallel to the x_3 axis, such that $p=(x_1^2+x_2^2)^{1/2}$, and the components of this force parallel to the x_i axes are

$$(\omega^2x_1,\ \omega^2x_2,\ 0).$$

The position vector $\boldsymbol{L}$ in (2.1.6) is now defined relative to the rotating frame. The gravitational potential W of the combined attraction and centrifugal force is

$$\boldsymbol{g}=\nabla W=\nabla(V+\tfrac{1}{2}\omega^2p^2). \tag{2.1.7}$$

where $\boldsymbol{g}$ is the gravity vector whose magnitude g is called gravity.

2.2. Elements of potential theory

THE LAPLACE AND POISSON EQUATIONS

The gravitational potential V is defined by eqn (2.1.5) and the force components, the first spatial derivatives of V, are defined by eqns (2.1.3) and (2.1.4). Outside of the body, the second derivates of V satisfy the condition,

$$\sum_i \partial^2 V/\partial x_i^2 \equiv \nabla^2 V = 0, \tag{2.2.1}$$

and this is *Laplace's equation*. Its solutions are called *harmonic functions* and the gravitational potential is a harmonic function outside of the body. Within the body this same operation results in

$$\nabla^2 V = -4\pi G\rho \tag{2.2.2}$$

and this is known as *Poisson's equation*. The operator $\nabla^2 = \Sigma_i(\partial^2/\partial x_i^2)$ is called the Laplacian operator. While V and ∇V are continuous across the boundary containing the mass, $\nabla^2 V$ is not, and neither Laplace's or Poisson's equation is valid at the boundary itself. The potential W of the combined gravity and centrifugal forces does not satisfy these equations. From (2.1.7)

$$\nabla^2 W = \nabla^2 V + \nabla^2(\tfrac{1}{2}\omega^2 p^2) = 2\omega^2 \tag{2.2.3}$$

outside the body, and

$$\nabla^2 W = -4\pi G\rho + 2\omega^2 \tag{2.2.4}$$

within the body. In terms of spherical coordinates r, ϕ, λ (ϕ is latitude, λ is longitude, both defined relative to the Earth-fixed frame x_i)

$$x_i = r(\cos\phi\cos\lambda, \cos\phi\sin\lambda, \sin\phi)$$

and Laplace's equation becomes

$$\nabla^2 V = \frac{\partial^2 V}{\partial r^2} + \frac{2}{r}\frac{\partial V}{\partial r} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} - \frac{\tan\phi}{r^2}\frac{\partial V}{\partial \phi} + \frac{1}{r^2\cos^2\phi}\frac{\partial^2 V}{\partial \lambda^2} = 0. \tag{2.2.5}$$

LEGENDRE POLYNOMIALS

It is mathematically convenient to expand the external potential into harmonic functions because such functions are also solutions of Laplace's equation. Spherical harmonics are particularly convenient for representing observations made on the surface of a sphere, or at the Earth's surface, and they facilitate the geophysical inversions of global data sets. If the angle between the two radius vectors $\boldsymbol{r}$ of the unit mass at $P(x_i)$ and $\boldsymbol{r}'$ of the mass element at $P'(x_i')$ is denoted by ψ (Fig. 2.1), then the distance L between these two points is

$$L = (r^2 + r'^2 - 2rr'\cos\psi)^{1/2}. \tag{2.2.6}$$

For $r > r'$ the reciprocal distance is

$$L^{-1} = r^{-1}\left[1 + \left(\frac{r'}{r}\right)^2 - 2\frac{r'}{r}\cos\psi\right]^{-1/2} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{r'}{r}\right)^n P_{n0}(\cos\psi), \tag{2.2.7a}$$

and for $r < r'$

$$L^{-1} = \frac{1}{r'} \sum_{n=0}^{\infty} \left(\frac{r}{r'}\right)^n P_{n0}(\cos\psi). \tag{2.2.7b}$$

The $P_{n0}(\cos\psi)$ are the conventional *Legendre polynomials* of degree n. They are defined by *Rodrigues*' formula, with $t = \cos\psi$, as (e.g. MacRobert 1967)

$$P_{n0}(t) = \frac{1}{2^n n!} \frac{\mathrm{d}^n}{\mathrm{d}t^n} (t^2 - 1)^n. \tag{2.2.8}$$

The zero order polynomials P_{n0} are called *zonal harmonics*. They have n distinct zeros between $\phi = \pi/2$ and $-\pi/2$ arranged symmetrically about $\phi = 0$, and for odd n the circle $\phi = 0$ forms one of this set. If the positions of P and P′ are expressed in spherical coordinates r, ϕ, λ then the geocentric angle ψ is given by

$$\cos\psi = \sin\phi \sin\phi' + \cos\phi \cos\phi' \cos(\lambda' - \lambda), \tag{2.2.9}$$

and substituting this into the above definition of the Legendre polynomial leads to the *addition theorem* (e.g. MacRobert 1967, p. 7),

$$P_{n0}(\cos\psi) = \sum_{m=0}^{n} (2 - \delta_{0m}) \frac{(n-m)!}{(n+m)!} P_{nm}(\sin\phi) \times P_{nm}(\sin\phi') \cos m(\lambda - \lambda') \tag{2.2.10}$$

where

$$\delta_{0m} = \begin{cases} 1 & \text{if } m = 0 \\ 0 & \text{if } m \neq 0 \end{cases}. \tag{2.2.11}$$

The $P_{nm}(\sin\phi)$ are the *associated Legendre polynomials* of degree n and order m. They are defined by, now with $t = \sin\phi$,

$$P_{nm}(t) = \frac{1}{2^n n!} (1 - t^2)^{m/2} \frac{\mathrm{d}^{n+m}}{\mathrm{d}t^{n+m}} (t^2 - 1)^n, \tag{2.2.12a}$$

or, alternatively, by

$$P_{nm}(t) = \frac{(1-t^2)^{m/2}}{2^n} \sum_{k=0}^{k^*} (-1)^k \frac{(2n-2k)!}{k!\,(n-k)!\,(n-m-2k)!} t^{(n-m-2k)} \tag{2.2.12b}$$

where k^* is the greatest integer $\leq (n-m)/2$ (Heiskanen and Moritz 1967, p. 24). The polynomials $P_{nm}(\sin\phi)(\cos m\lambda$ or $\sin m\lambda)$ with $0 < m < n$ are caled *tesseral harmonics*. These functions have zeros along n–m circles whose pole is $\phi = \pi/2$ and along m equally spaced great circles passing through $\phi = \pi/2$. For $m = n$ the polynomials are called *sectorial harmonics* but frequently the name tesseral harmonics is used to include all

Table 2.1

Unnormalized Legendre and associated Legendre functions $P_{nm}(\sin\phi)$ of degree n, order m.

	$m=0$	$m=1$	$m=2$	$m=3$
$n=0$	1			
$n=1$	$\sin\phi$	$\cos\phi$		
$n=2$	$\frac{3}{2}\sin^2\phi-\frac{1}{2}$	$3\sin\phi\cos\phi$	$3\cos^2\phi$	
$n=3$	$\frac{5}{2}\sin^3\phi-\frac{3}{2}\sin\phi$	$\cos\phi(\frac{15}{2}\sin^2\phi-\frac{3}{2})$	$15\cos^2\phi\sin\phi$	$15\cos^3\phi$

$m\neq 0$ harmonics, irrespective of their order. Table 2.1 summarizes some of the low degree and order functions.

With (2.2.7a) and (2.2.10) the potential V of (2.1.5) at $r>R$ can be expanded into spherical harmonics as

$$V=\frac{G\mathcal{M}}{r}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{R_e}{r}\right)^n(C_{nm}\cos m\lambda+S_{nm}\sin m\lambda)P_{nm}(\sin\phi) \tag{2.2.13}$$

where

$$\left.\begin{matrix}C_{nm}\\S_{nm}\end{matrix}\right\}=\frac{1}{\mathcal{M}R_e^n}(2-\delta_{0m})\frac{(n-m)!}{(n+m)!}\int_{\mathcal{M}}(r')^nP_{nm}(\sin\phi')\left\{\begin{matrix}\cos m\lambda'\\\sin m\lambda'\end{matrix}\right\}\mathrm{d}\mathcal{M}. \tag{2.2.14a}$$

The C_{nm}, S_{nm} are the *Stokes coefficients*. R_e refers here to the equatorial radius of the planet. Sometimes the mean radius R is used in eqn (2.2.13) instead of R_e and in this case the definition (2.2.14a) of the coefficients must be correspondingly modified. That is,

$$\left.\begin{matrix}C_{nm}\\S_{nm}\end{matrix}\right\}=\frac{1}{\mathcal{M}R^n}(2-\delta_{0m})\frac{(n-m)!}{(n+m)!}\int_{\mathcal{M}}(r')^nP_{nm}(\sin\phi')\left\{\begin{matrix}\cos m\lambda'\\\sin m\lambda'\end{matrix}\right\}\mathrm{d}\mathcal{M}. \tag{2.2.14b}$$

With the abbreviations

$$Y_{inm}=P_{nm}(\sin\phi)\begin{cases}\cos m\lambda & \text{if } i=1\\ \sin m\lambda & \text{if } i=2\end{cases} \tag{2.2.15a}$$

$$C_{inm}=\begin{cases}C_{nm} & \text{if } i=1\\ S_{nm} & \text{if } i=2\end{cases}, \tag{2.2.15b}$$

the potential is written in the abbreviated form

$$V(r,\phi,\lambda)=\frac{G\mathcal{M}}{r}\sum_{i=1}^{2}\sum_{n=0}^{\infty}\sum_{m=0}^{n}\left(\frac{R}{r}\right)^n C_{inm}Y_{inm}. \tag{2.2.15c}$$

SOME PROPERTIES OF LEGENDRE POLYNOMIALS

The Legendre polynomials have several important properties, foremost of which are the *orthogonality relations*,

$$\int_S Y_{inm} Y_{jpq}\, \mathrm{d}S = 0 \tag{2.2.16a}$$

when $i \neq j$, $n \neq p$ or $m \neq q$, and

$$\int_S [Y_{inm}]^2\, \mathrm{d}S = 4\pi/\Pi_{nm}^2 \tag{2.2.16b}$$

where the integrals are over the surface S of a sphere of unit radius (e.g. MacRobert 1967). The normalizing factor Π_{nm} is defined by

$$\Pi_{nm}^2 = (2 - \delta_{0m})(2n+1)(n-m)!/(n+m)!. \tag{2.2.16c}$$

The definition (2.2.12) of the Legendre polynomials corresponds to the *unnormalized* functions frequently used in theoretical expansions of the potential. This usage does have the numerical disadvantage that as the degree and order increases, the term $(n+m)!$ in the denominators of eqn (2.2.14) becomes increasingly larger and to avoid this, *normalized* polynomials $\bar{Y}_{inm}$ are sometimes introduced. These are defined such that

$$\int_S [\bar{Y}_{inm}]^2\, \mathrm{d}S = 4\pi, \tag{2.2.16d}$$

or

$$\bar{P}_{inm} = \Pi_{nm} P_{inm}$$

and

$$\bar{Y}_{inm} = \Pi_{nm} Y_{inm}. \tag{2.2.17a}$$

The corresponding *normalized Stokes coefficients* (2.2.14) are

$$\bar{C}_{inm} = \frac{1}{\Pi_{nm}} C_{inm}. \tag{2.2.17b}$$

The Y_{inm} defined by (2.2.15a) are *surface harmonics* and the products (r^n or $r^{-(n+1)}$)Y_{inm} are referred to as *solid spherical harmonics*. The latter are solutions of Laplace's equation, as is readily verified by substituting them into eqn (2.2.5). It also follows from (2.2.5) that

$$\left(\frac{\partial^2}{\partial\phi^2} - \tan\phi\frac{\partial}{\partial\phi} + \frac{1}{\cos^2\phi}\frac{\partial^2}{\partial\lambda^2}\right) Y_{inm} = -n(n+1)Y_{inm}. \tag{2.2.18}$$

In some problems certain infinite sums of Legendre polynomials occur which can be expressed by analytical functions. These include

(Hobson 1932; Farrell 1972);

$$\left.\begin{aligned}
\sum_{n=0}^{\infty} P_n(\cos\psi) &= \frac{1}{2\sin(\psi/2)}\\
\sum_{n=0}^{\infty} nP_n(\cos\psi) &= \frac{-1}{4\sin(\psi/2)}\\
\sum_{n=1}^{\infty} \frac{\partial P_n(\cos\psi)}{\partial\psi} &= \frac{-\cos(\psi/2)}{4\sin^2(\psi/2)}\\
\sum_{n=1}^{\infty} \frac{1}{n}\frac{\partial P_n(\cos\psi)}{\partial\psi} &= -\frac{\cos(\psi/2)[1+2\sin(\psi/2)]}{2\sin(\psi/2)[1+\sin(\psi/2)]}.
\end{aligned}\right\} \tag{2.2.19}$$

STOKES COEFFICIENTS

The Stokes coefficients (2.2.14a) represent integrals of functions of the mass distribution within the planet. For zero order, the S_{n0} vanish and the remaining coefficients C_{n0} are referred to as zonal coefficients. For degree 0

$$C_{00} = \frac{1}{\mathcal{M}R_e}\int_{\mathcal{M}} r'\,\mathrm{d}\mathcal{M} = 1$$

and the first term in the potential (2.2.13) is simply $G\mathcal{M}/r$, the potential at r caused by a radially symmetric sphere of mass $\mathcal{M}$. For degree 1,

$$\begin{aligned}
C_{10} &= \frac{1}{\mathcal{M}R_e}\int_{\mathcal{M}} r'\sin\phi'\,\mathrm{d}\mathcal{M} \equiv \frac{1}{\mathcal{M}R_e}\int_{\mathcal{M}} x_3'\,\mathrm{d}\mathcal{M}\\
C_{11} &= \frac{1}{\mathcal{M}R_e}\int_{\mathcal{M}} r'\cos\phi'\cos\lambda'\,\mathrm{d}\mathcal{M} \equiv \frac{1}{\mathcal{M}R_e}\int_{\mathcal{M}} x_1'\,\mathrm{d}\mathcal{M}\\
S_{11} &= \frac{1}{\mathcal{M}R_e}\int_{\mathcal{M}} r'\cos\phi'\sin\lambda'\,\mathrm{d}\mathcal{M} \equiv \frac{1}{\mathcal{M}R_e}\int_{\mathcal{M}} x_2'\,\mathrm{d}\mathcal{M}
\end{aligned} \tag{2.2.20}$$

and these three coefficients represent the coordinates of the centre of mass of the body (normalized by R_e). They vanish if the origin of the coordinate system x_i is located at the centre of mass. The potential (2.2.13) can therefore be written as $V = V_0 + \Delta V$ where $V_0 = G\mathcal{M}/r$ and

$$\Delta V(r,\phi,\lambda) = \frac{G\mathcal{M}}{r}\sum_{i=1}^{2}\sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{R_e}{r}\right)^n C_{inm}Y_{inm}. \tag{2.2.21a}$$

The second degree zonal Stokes coefficients follow from (2.2.14) and Table 2.1 as

$$C_{20} = \frac{-1}{\mathcal{M}R_e^2}[I_{33} - \tfrac{1}{2}(I_{11}+I_{22})], \tag{2.2.22a}$$

where

$$I_{ii} = \int_{\mathcal{M}} (x_{i+1}^2 + x_{i+2}^2)\, \mathrm{d}\mathcal{M} \quad (i = 1, 2, 3) \qquad (2.2.23a)$$

represent the *moments of inertia* with respect to the x_i axes. Likewise

$$\begin{aligned} C_{21} &= \frac{I_{13}}{\mathcal{M}R_e^2}, & S_{21} &= \frac{I_{23}}{\mathcal{M}R_e^2}, \\ C_{22} &= \frac{1}{4\mathcal{M}R_e^2}(I_{22} - I_{11}), & S_{22} &= \frac{I_{12}}{2\mathcal{M}R_e^2}, \end{aligned} \qquad (2.2.22b)$$

with

$$I_{ij} = \int_{\mathcal{M}} x_i x_j \, \mathrm{d}\mathcal{M}. \qquad (2.2.23b)$$

If R is used in (2.2.13) instead of R_e then the above expressions (2.2.20) to (2.2.23) must be modified accordingly.

To a good approximation the mean position of the Earth's rotation axis lies close to the mean position of the axis of maximum inertia and x_3 lies close to a principal axis. The $I_{13}/\mathcal{M}R_e^2$ and $I_{23}/\mathcal{M}R_e^2$ will therefore be small quantities in most cases and the corresponding potential to degree 2 is

$$\begin{aligned} V_2 \simeq \frac{G\mathcal{M}}{r} &+ \frac{G}{2r^3}[I_{33} - \tfrac{1}{2}(I_{11} + I_{22})](1 - 3\sin^2\phi) \\ &+ \frac{3}{4}\frac{G}{r^3}[(I_{22} - I_{11})\cos 2\lambda + I_{12}\sin 2\lambda]\cos\phi. \end{aligned} \qquad (2.2.21b)$$

Theoretical considerations of a rotating, fluid-like body indicates that the density distribution of the body will be symmetrical about the rotation axis so that $I_{12}/\mathcal{M}R_e^2$ and $(I_{11} - I_{22})/\mathcal{M}R_e^2$ can also be expected to be small quantities. The dominant Stokes coefficient of degree 2 will then be C_{20}, a measure of the Earth's flattening and which, for a fluid with the same mass, density distribution, and angular velocity as the Earth, will be of the order 10^{-3} (see below). This is indeed observed for the Earth, with (Gaposchkin 1977; Lerch *et al.* 1979)

$$\left.\begin{aligned} C_{20} &= -1082.63 \times 10^{-6} \\ C_{21}, S_{21} &= \mathcal{O}(10^{-9}) \\ C_{22} &= 1.57 \times 10^{-6}, \qquad S_{22} = -0.90 \times 10^{-6}. \end{aligned}\right\} \qquad (2.2.24)$$

$\mathcal{O}(x)$ refers to terms of quantities of the order of magnitude of x. These second degree Stokes coefficients are defined here with respect to the mean equatorial radius R_e (eqn 2.2.14a) rather than the mean radius R. All other coefficients are of the order $(C_{20})^2$ or smaller. For slowly rotating planets such as Venus, Mercury, or the Moon, C_{20} is considerably smaller and it need not necessarily be the dominant term in the corresponding gravitational potential expansion.

THE GEOID

Surfaces of constant gravitational potential, $W(x) = \text{constant} = W_0$, are called *equipotential surfaces* or *level surfaces*. The difference in potential dW between two nearby points separated by a distance dx is

$$\mathrm{d}W = \sum_i \frac{\partial W}{\partial x_i} \mathrm{d}x_i = \nabla W \cdot \mathrm{d}\mathbf{x} = \mathbf{g} \cdot \mathrm{d}\mathbf{x}$$

and if the vector d$\mathbf{x}$ lies along W_0 then $\mathrm{d}W = \mathbf{g} \cdot \mathrm{d}\mathbf{x} = 0$. The gravity vector $\mathbf{g}$ is therefore orthogonal to the equipotential surface passing through the same point and plumb lines or verticals are perpendicular to the level surfaces that they intersect.

In the absence of dynamical forces (winds, currents, for example), the ocean surface is a level surface of potential W_0. The ocean, therefore, provides a natural definition for the shape of the Earth, in particular as a number of geodetic measurements relate directly to this surface. Heights, measured by spirit levelling, are measured relative to the geoid (Chapter 5) and the radar altimeter measurements of the sea surface from satellites provides a nearly direct estimate of the shape of this surface (Chapter 6). This equipotential surface is called the *geoid*. It will lie partly within the Earth and the surface has to be extended mathematically to the continental areas. There the geoid does not have a unique definition and it is a function of the density distribution within the crust (Chapter 5).

Outside the Earth the equipotential surfaces are everywhere defined. Lines intersecting these surfaces perpendicularly specify the direction of the gravity vector, or the vertical or direction of the plumb line. Heights measured along these verticals with respect to the geoid are the *orthometric heights* discussed further in Chapter 5.

REFERENCE ELLIPSOID

The gravitational potential W at P follows from (2.1.7) and (2.2.13) as

$$W(r, \phi, \lambda) = \frac{G\mathcal{M}}{r}\left[1 + \sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{R_e}{r}\right)^n (C_{nm}\cos m\lambda + S_{nm}\sin m\lambda)P_{nm}(\sin\phi)\right] + \tfrac{1}{3}\omega^2 r^2[1 - P_{20}(\sin\phi)] \qquad (2.2.25a)$$

because the distance of P from the rotation axis is

$$p^2 = r^2\cos^2\phi = \tfrac{2}{3}r^2[1 - P_{20}(\sin\phi)].$$

For the Earth C_{20} is the dominant term in the potential and a first approximation, U, of W is, at $r > R$,

$$U(r, \phi) = \frac{G\mathcal{M}}{r}\left\{1 + \frac{1}{3}\frac{\omega^2 r}{g_0(r)} + \left[C_{20}\left[\frac{R_e}{r}\right]^2 - \frac{1}{3}\frac{\omega^2 r}{g_0(r)}\right]P_{20}(\sin\phi)\right\} \tag{2.2.25b}$$

where $g_0(r) = G\mathcal{M}/r^2$. The shape of the equipotential surface corresponding to U is a function of latitude only and can be written in the form

$$r(\phi) = R[1 - \tfrac{2}{3}fP_{20}(\sin\phi)] + \mathcal{O}(f^2) \tag{2.2.26a}$$

with $f = (R_e - R_p)/R_e$ where R_e and R_p are the equatorial and polar radii and R is the mean radius. Equation (2.2.26a) represents the equation for an ellipsoid of revolution and, as a first approximation, the Earth's shape can be approximated by such a figure whose short axis coincides with the rotation axis. A relation between f and C_{20} follows by evaluating U at the equator $U(r = R_e, \phi = 0)$ with U at the pole $U(r = R_p, \phi = \pi/2)$, the two values being, by definition, on the same equipotential surface. The result is

$$-C_{20} = \tfrac{2}{3}f(1 - \tfrac{1}{2}f) - \tfrac{1}{3}m(1 - \tfrac{3}{2}m - \tfrac{2}{7}f) + \mathcal{O}(f^3) \tag{2.2.26b}$$

where

$$m = \frac{\text{centrifugal force at the equator}}{\text{gravity at the equator}} = \frac{\omega^2 R_e}{\gamma_e} \simeq 3 \times 10^{-3}.$$

The mean radius R in (2.2.26a) relates to the equatorial radius R_e by

$$R = R_e\left(1 - \frac{f}{3} - \frac{f^2}{5} + \ldots\right) \tag{2.2.27a}$$

and the theoretical gravity at the equator is

$$\gamma_e = \frac{G\mathcal{M}}{R_e^2}(1 - f + \tfrac{3}{2}m - \tfrac{15}{14}mf)^{-1}. \tag{2.2.27b}$$

The theoretical gravity at any latitude ϕ is given by

$$\gamma = \gamma_e(1 + f_2\sin^2\phi - \tfrac{1}{4}f_4\sin^2 2\phi + \ldots) \tag{2.2.27c}$$

with

$$\begin{aligned} f_2 &= -f + \tfrac{5}{2}m - \tfrac{17}{14}fm + \tfrac{15}{4}m^2 + \ldots \\ f_4 &= -\frac{f^2}{2} + \tfrac{5}{2}fm + \ldots \end{aligned} \tag{2.2.27d}$$

The theoretical gravity at small heights above the ellipsoid, γ_h, can be expanded as

$$\gamma_h = \gamma + \frac{\partial\gamma}{\partial h}h + \frac{1}{2}\frac{\partial^2\gamma}{\partial h^2}h^2 + \ldots$$

and with (2.2.27c)

$$\gamma_h = \gamma - \frac{2\gamma_e}{R_e}[1 + f + m + (-3f + \tfrac{5}{2}m)\sin^2\phi]h + \frac{3\gamma_e}{R_e^2}h^2. \quad (2.2.27e)$$

Together, eqns (2.2.25), (2.2.26a), and (2.2.27) define the potential, shape, and gravity of the first-order approximation of the Earth (see Heiskanen and Moritz 1967, for details). These definitions involve four parameters f, R_e, γ_e, and ω that are determined from the geodetic observations of the shape, gravity, and rotation of the planet. They provide a convenient reference surface with respect to which all departures in geometrical shape or in gravity can be treated as small quantities. Departures $\mathcal{N}$ of the observed equipotential from the theoretical equipotential surface may amount to 100 m or more and this quantity can be measured with a precision $\sigma_{\mathcal{N}}$ approaching 10 cm. Thus $\sigma_{\mathcal{N}}/R \simeq 10^{-8}$, less than quantities of the order f^2 or fm. The Stokes coefficient C_{20} is measured with a similar precision, as is gravity, and a more complete theory for the reference ellipsoid relations must contain higher order terms, of the order f^3 and f^4. Also, relations such as (2.2.25b) must contain terms in C_{40} and C_{60} (see Hirvonen 1960; Lambert 1961; Heiskanen and Moritz 1967).

Geodetic practice is to adopt a set of parameters that gives the best ellipsoidal approximation to the geoid and whose potential at its surface equals that of the geoid. This choice minimizes the departures of the observed quantities such as g from the theoretical values and reduces the number of higher order terms in the theoretical expressions such as (2.2.27). Recent observations and analyses lead to the following values for the fundamental geodetic parameters (EOS 1983).

$$\begin{aligned}
G\mathcal{M} &= (39\,860\,044 \pm 1)10^7\ \mathrm{m^3\,s^{-2}}\\
C_{20} &= -(1\,082\,629 \pm 1)10^{-9}\\
R_e &= (6\,378\,136 \pm 1)\ \mathrm{m}\\
f^{-1} &= 298.275 \pm 0.001\\
\omega &= 7\,292\,115 \times 10^{-11}\ \mathrm{rad\,s^{-1}}\\
\gamma_e &= (978\,032 \pm 1)10^{-5}\ \mathrm{m\,s^{-2}}\\
m &= 0.00345.
\end{aligned} \quad (2.2.28)$$

HYDROSTATIC EQUILIBRIUM

The best-fitting ellipsoidal approximation of the geoid has no physical meaning and a geophysically more useful reference figure is the hydrostatic equilibrium shape of a body whose mass, radial density distribution and rotation are the same as the observed values for the planet.

Departures from this idealized shape can, therefore, be attributed to departures from the hydrostatic stress state of the planet and observations of the geoid height $\mathcal{N}$, for example, can be related, albeit not uniquely, to deviatoric stresses inside the body. The first-order solution for the flattening f_{he} of this ellipsoid is (Jeffreys 1970; Bullen 1975)

$$f_{he} = \frac{5}{2}\frac{\omega^2 R_e}{\gamma_e}\left\{1 + \left(\frac{5}{2} - \frac{15}{4}\frac{I_{33}}{\mathcal{M}R^2}\right)^2\right\}^{-1} + \mathcal{O}(f^2) \qquad (2.2.29a)$$

and the density distribution of the planet enters only through the moment of inertia I_{33}. This particular solution is based on the Radau transformation which provides a good approximation for the Earth and other terrestrial planets. The moment of inertia is deduced from the precession constant H (2.4.20b, see below) or, for $I_{11} \neq I_{22}$,

$$H = [I_{33} - \tfrac{1}{2}(I_{11} + I_{22})]/I_{33} \qquad (2.2.29b)$$

and, with the definition (2.2.22a) of the second degree Stokes coefficient,

$$\frac{I_{33}}{\mathcal{M}R_e^2} = \frac{-C_{20}}{H} = 0.331. \qquad (2.2.29c)$$

For the Earth, the departures from hydrostatic equilibrium are of the order f^2 and a higher accuracy theory, which does require a more detailed consideration of the density structure of the planet, is necessary. Nakiboglu (1982) obtained

$$f_{he} = 1/299.638, \qquad (2.2.30a)$$

and with (2.2.26b) and the value of m given by (2.2.28)

$$C_{20\,|he} = -1.072618 \times 10^{-3}. \qquad (2.2.30b)$$

Also

$$C_{40\,|he} = -2.992 \times 10^{-6}.$$

GRAVITY AND GEOID ANOMALIES

Gravity measured on the surface of the Earth varies considerably from place to place because of variations in the distance to the centre of mass of the Earth, because of the attraction of nearby topography and because of lateral variations in density within the crust and mantle. To minimize these variations and to separate the last cause from the other more mundane contributions, it is appropriate to reduce the observations to the geoid before comparing them with the reference gravity. This reduction comprises the *free-air correction* that takes into account the variation of gravity with the height of the measurement point P′ above the geoid, and the *Bouguer correction* that takes into account the

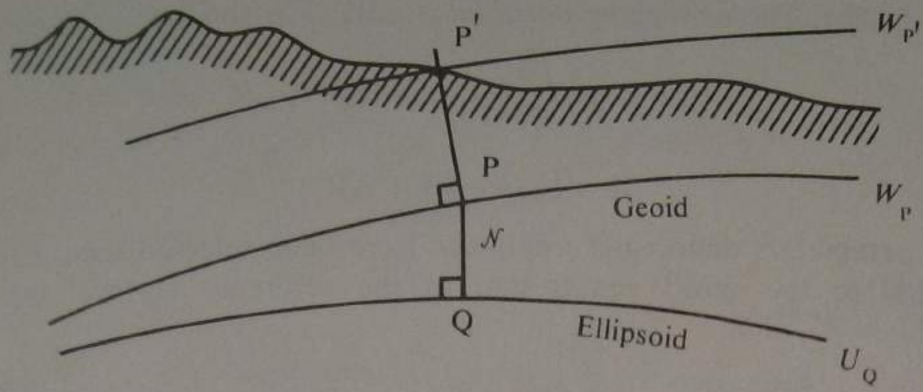

FIG. 2.2. Definition of geoid height $\mathcal{N}$. Gravity is observed on the Earth's surface at the point P′ on an equipotential surface $W_{P'}$. P is the projection of P′, along the vertical through P′, onto the geoid of potential W_P and Q is the projection of P onto the reference ellipsoid, along the normal to the ellipsoid, of potential U_Q. The geoid height $\mathcal{N}$ of P is given by the distance PQ.

attraction of the topography between the geoid and P′. These corrections are discussed in Chapter 5 and are here assumed to have been carried out. Figure 2.2 illustrates the geometry. Gravity, measured at P′ on the surface, is projected along the vertical to a point P on the geoid. The projection of P onto the reference ellipsoid, along the normal to this ellipsoid, is at Q and the distance from P to Q is the *geoid height* $\mathcal{N}$.

Gravity observations, g_P, reduced to the geoid, can be compared with the theoretical gravity according to

$$\delta g_P = g_P - \gamma_P. \qquad (2.2.31a)$$

This is the *gravity perturbation*, but its evaluation requires a knowledge of $\mathcal{N}$ because

$$\gamma_P = \gamma_Q + \frac{\partial \gamma}{\partial r}\mathcal{N},$$

where γ_Q is the theoretical gravity on the ellipsoid and is given by (2.2.27c). A more useful definition is the *gravity anomaly*, which is independent of $\mathcal{N}$, and is defined as

$$\Delta g_P = g_P - \gamma_Q = \delta g_P + (\gamma_P - \gamma_Q). \qquad (2.2.32a)$$

To relate the gravity anomaly and geoid height to the potential, W_P at P can be written as

$$W_P = U_P + \Delta W_P,$$

where U_P is the potential of the reference ellipsoid evaluated at P (e.g.

eqn 2.2.25b). The *perturbing potential* is ΔW_P. For small $\mathcal{N}$

$$U_P = U_Q + \left(\frac{\partial U}{\partial r}\right)_Q \mathcal{N} = U_Q - \gamma_Q \mathcal{N}$$

and

$$W_P = U_Q - \gamma_Q \mathcal{N} + \Delta W_P.$$

If the parameters defining the ellipsoid have been selected such that the potential of the geoid equals that of the reference surface, that is, $W_P = U_Q \equiv W_0$, then

$$\mathcal{N} = \Delta W_P / \gamma_Q. \qquad (2.2.33)$$

This is *Brun's formula* which relates the shape of the geoid $\mathcal{N}$ to the perturbing potential ΔW (e.g. Heiskanen and Moritz 1967).

With $g_P = -(\partial W/\partial r)_P$ and $\gamma_P = -(\partial U/\partial r)_P$, the gravity perturbation is

$$\delta g_P = -\frac{\partial}{\partial r}(W - U)_P = -\frac{\partial \Delta W_P}{\partial r}, \qquad (2.2.31b)$$

and the gravity anomaly is

$$\Delta g_P = -\frac{\partial}{\partial r}\Delta W_P + \frac{\partial \gamma}{\partial r}\mathcal{N} = -\frac{\partial}{\partial r}\Delta W_P - \frac{2\Delta W_P}{r}. \qquad (2.2.32b)$$

In terms of the spherical harmonic expansion (2.2.15), the perturbing potential ΔW, geoid height $\mathcal{N}$, and gravity anomaly Δg become

$$\Delta W = \frac{G\mathcal{M}}{r}\sum_{i=1}^{2}\sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{R_e}{r}\right)^n Y_{inm} C^*_{inm}, \qquad (2.2.34a)$$

$$\mathcal{N} = R_e \sum_{i=1}^{2}\sum_{n=2}^{\infty}\sum_{m=0}^{n} Y_{inm} C^*_{inm}, \qquad (2.2.34b)$$

$$\Delta g = \gamma_e \sum_{i=1}^{2}\sum_{n=2}^{\infty}\sum_{m=0}^{n} (n-1) Y_{inm} C^*_{inm}, \qquad (2.2.34c)$$

where $C^*_{inm} = C_{inm}$ unless $i = 1$, n is even and $m = 0$, for then

$$C^*_{1n0} = (C_{1n0})_{\text{observed}} - (C_{1n0})_{\text{reference}}. \qquad (2.3.34d)$$

STOKES' INTEGRAL

An explicit relation between gravity anomalies and geoid height is given by *Stokes' integral*

$$\mathcal{N}(P) = \frac{R}{4\pi G}\int_S \Delta g(P') S^*(\psi_{PP'})\, dS, \qquad (2.2.35)$$

where $S^*(\psi_{PP'})$ is a weighting function, dependent on the geocentric eqn 2.2.25b). The *perturbing potential* is ΔW_P. For small $\mathcal{N}$

$$U_P = U_Q + \left(\frac{\partial U}{\partial r}\right)_Q \mathcal{N} = U_Q - \gamma_Q \mathcal{N}$$

and

$$W_P = U_Q - \gamma_Q \mathcal{N} + \Delta W_P.$$

If the parameters defining the ellipsoid have been selected such that the potential of the geoid equals that of the reference surface, that is, $W_P = U_Q \equiv W_0$, then

$$\mathcal{N} = \Delta W_P / \gamma_Q. \tag{2.2.33}$$

This is *Brun's formula* which relates the shape of the geoid $\mathcal{N}$ to the perturbing potential ΔW (e.g. Heiskanen and Moritz 1967).

With $g_P = -(\partial W/\partial r)_P$ and $\gamma_P = -(\partial U/\partial r)_P$, the gravity perturbation is

$$\delta g_P = -\frac{\partial}{\partial r}(W - U)_P = -\frac{\partial \Delta W_P}{\partial r}, \tag{2.2.31b}$$

and the gravity anomaly is

$$\Delta g_P = -\frac{\partial}{\partial r}\Delta W_P + \frac{\partial \gamma}{\partial r}\mathcal{N} = -\frac{\partial}{\partial r}\Delta W_P - \frac{2\Delta W_P}{r}. \tag{2.2.32b}$$

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$$\Delta W = \frac{G\mathcal{M}}{r}\sum_{i=1}^{2}\sum_{n=2}^{\infty}\sum_{m=0}^{n}\left(\frac{R_e}{r}\right)^n Y_{inm} C^*_{inm}, \tag{2.2.34a}$$

$$\mathcal{N} = R_e \sum_{i=1}^{2}\sum_{n=2}^{\infty}\sum_{m=0}^{n} Y_{inm} C^*_{inm}, \tag{2.2.34b}$$

$$\Delta g = \gamma_e \sum_{i=1}^{2}\sum_{n=2}^{\infty}\sum_{m=0}^{n} (n-1) Y_{inm} C^*_{inm}, \tag{2.2.34c}$$

where $C^*_{inm} = C_{inm}$ unless $i = 1$, n is even and $m = 0$, for then

$$C^*_{1n0} = (C_{1n0})_{\text{observed}} - (C_{1n0})_{\text{reference}}. \tag{2.3.34d}$$

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where $S^*(\psi_{PP'})$ is a weighting function, dependent on the geocentric angle ψ subtended by the point P at which $\mathcal{N}$ is evaluated and the moving point P′ at which Δg is given (Heiskanen and Vening Meinesz 1958). This function does not converge rapidly and gravity must be specified over the entire surface of the sphere if an unbiased estimate of $\mathcal{N}$ is sought. If Δg is known over only part of the sphere, S', the integral adequately reflects only the shorter wavelength variations in $\mathcal{N}$, those of wavelengths less than about the average linear dimension of the area S'.

The integral (2.2.35) is approximate only and rests on several assumptions. The first assumption is that no mass lies outside the geoid on which the gravity anomalies are defined and this requires that a number of elaborate corrections to the gravity observations are made (see section 5.2). A second assumption is the implied spherical approximation, in which the integral (2.2.35) is taken over a sphere and not over the reference ellipsoid. The error so introduced is of the order $f\mathcal{N}$, or about 30 cm for $\mathcal{N} = 100$ m. A third assumption is that the reference surface has the same potential as the geoid, that this surface encloses a mass equal to that of the Earth, and that this surface has its origin at the centre of mass of the Earth. If these conditions are not met then corrective terms must be added to eqn (2.2.35) if results at the sub-metre level of precision are sought (see Hirvonen 1960; Molodenskiy *et al.* 1962; Heiskanen and Moritz 1967, for more detailed discussions of the Stokes integral).

SPHERICAL HARMONIC EXPANSIONS

Any single-valued function on a sphere can be expanded into a series of surface harmonics Y_{inm} defined by (2.2.15a). For example, gravity anomalies on the geoid can be expanded as

$$\Delta g(R, \phi, \lambda) = \gamma_e \sum_{i=1}^{2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} g_{inm} Y_{inm}, \qquad (2.2.36a)$$

where the g_{inm} represent a series of dimensionless coefficients. Multiplying both sides by a polynomial $Y_{i'n'm'}$ and integrating the products over a sphere, results in a non-zero integral only for those terms for which $i = i'$, $n = n'$, $m = m'$ (eqn 2.2.16). Hence

$$g_{inm} = \frac{\Pi_{nm}^2}{4\pi\gamma_e} \int_S \Delta g(R, \phi, \lambda) Y_{inm} \, dS \qquad (2.2.36b)$$

where the normalizing factor Π_{nm}^2 is given by (2.2.16c). Comparing the expansion (2.2.36a) with (2.2.34c) leads to

$$C_{inm}^* = \frac{\Pi_{nm}^2}{4\pi\gamma_e(n-1)} \int_S \Delta g(R, \phi, \lambda) Y_{inm} \, dS \qquad (2.2.37)$$

Table 2.2

Low degree and order coefficients in the ocean function $O(\phi, \lambda)$ defined by eq (2.2.38d). These coefficients are a subset of an expansion to degree and order 1 based on a 10′ resolution of the global coastline.

n	O_{1n0}	O_{1n1}	O_{1n2}	O_{1n2}	O_{2n2}	O_{1n3}	O_{2n3}	O_{1n4}	O_{2n4}
0	0.699								
1	−0.128	−0.109	−0.062						
2	−0.056	−0.043	−0.061	0.044	0.002				
3	0.044	0.042	−0.036	0.066	−0.092	−0.012	−0.087		
4	−0.026	0.038	0.027	0.086	−0.027	−0.050	0.003	0.018	−0.10

and the Stokes coefficients can be evaluated directly from the gravity anomalies.

Other geophysical quantities measured on the Earth's surface can be expanded likewise into surface harmonics. For elevations $h(R, \phi, \lambda)$ above the geoid, for example,

$$h(R, \phi, \lambda) = R \sum_{i=1}^{2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} h_{inm} Y_{inm}, \qquad (2.2.38a)$$

where the dimensionless topographic coefficients h_{inm} are given by the integrals

$$h_{inm} = \frac{\Pi_{nm}^2}{4\pi R} \int_S h(R, \phi, \lambda) Y_{inm} \, dS. \qquad (2.2.38b)$$

Seismic velocity anomalies δv, which vary with depth throughout the mantle, may be expanded as

$$\delta v(r, \phi, \lambda) = \sum_{k=0}^{K} {}_k v_0 \sum_{n=0}^{\infty} \sum_{m=0}^{n} f_k(r) {}_k v_{inm} Y_{inm}, \qquad (2.2.38c)$$

where $f_k(r)$ is a function of radial distance (e.g. Dziewonski 1984) and where ${}_k v_{inm}$ are dimensionless coefficients specifying the departure of the seismic velocity distribution from the mean value ${}_k v_0$. Another quantity that is usefully expanded into spherical harmonics is the *ocean function* $O(\phi, \lambda)$ defined as unity where there are oceans and zero where there is land. Then

$$O(\phi, \lambda) = \sum_{i=1}^{2} \sum_{n=0}^{\infty} \sum_{m=0}^{n} O_{inm} Y_{inm}. \qquad (2.2.38d)$$

Low degree and order coefficients O_{inm} are given in Table 2.2.

POWER SPECTRA

Information on the wavelength characteristics of a particular field observed on the Earth's surface is summarized in its power spectrum. If the quantity, for example the perturbing potential ΔW, is expressed in fully normalized spherical harmonics, the discrete dimensionless power spectrum is defined as (Kaula 1967a; Heiskanen and Moritz 1967),

$$\mathcal{V}_n^2\{\Delta W\} = \sum_i \sum_m \bar{C}_{inm}^2. \tag{2.2.39a}$$

Likewise, the dimensionless power spectrum of the topography (2.2.38a) is

$$\mathcal{V}_n^2\{h\} = \sum_i \sum_m \bar{h}_{inm}^2 \tag{2.2.39b}$$

where the $\bar{h}_{inm}$ are fully normalized coefficients. That of the gravity anomalies is

$$\mathcal{V}_n^2\{\Delta g\} = \gamma_e^2(n-1)^2 \mathcal{V}_n^2\{\Delta W\} = \gamma_e^2(n-1)^2 \sum_i \sum_m \bar{C}_{inm}^2. \tag{2.2.39c}$$

The fully normalized coefficients are used throughout in these definitions. A measure of the correlation between two quantities, for example, ΔW and h, is given by the discrete cross-spectrum as

$$\mathcal{V}_n^2\{\Delta W, h\} = \left[\sum_i \sum_m \bar{C}_{inm}\bar{h}_{inm}\right][\mathcal{V}_n^2\{\Delta W\}\mathcal{V}_n^2\{h\}]^{-1/2}. \tag{2.2.40}$$

DIRICHLET'S PROBLEM

A frequently encountered problem is one in which gravity or potential is required on some surface other than the one on which the original measurements were made. This is a problem of upwards and downwards continuation of a potential field. The upwards continuation problem, in which the potential is computed at a greater distance from the anomalous mass than at which it is observed, is stable and presents little difficulty. In contrast, the reverse problem of downwards continuation is intrinsically unstable. An example of the former is the computation of the gravitational potential at satellite heights from gravity measurements taken at ground level. An example of the downward continuation is the computation of the shape of the geoid from the perturbing potential measured at satellite heights.

Let the perturbing potential due to a body of mass $\mathcal{M}$ be given on a surface $r = R^*$ as (cf. 2.2.21a)

$$\Delta V(R^*, \phi, \lambda) = \frac{G\mathcal{M}}{R^*} \sum_i \sum_n \sum_m C_{inm} Y_{inm}.$$

The potential inside or outside this surface is determined by a function $\Delta V(r, \phi, \lambda)$ that equals $\Delta V(R^*, \phi, \lambda)$ on R^* and which also satisfies

Laplace's equation (2.2.5) outside the body. The determination of the potential is known as *Dirichlet's first boundary-value problem* of potential theory (e.g. Kellogg 1953; Sternberg and Smith 1964). These conditions are satisfied by the solid spherical harmonics

$$\left(\frac{G\mathcal{M}}{R^*}\right)\left(\frac{R^*}{r}\right)^{n+1} C_{inm} Y_{inm} \quad \text{and} \quad \left(\frac{G\mathcal{M}}{R^*}\right)\left(\frac{r}{R^*}\right)^{n} C_{inm} Y_{inm}.$$

Outside the surface $r = R^*$ the potential must vanish as r goes to infinity and only the first of these two forms is appropriate at $r > R^*$. Inside the surface, $r < R^*$, the potential must be regular at the origin and now the second of the above two forms is appropriate. Then

$$\Delta V(r > R^*) = \sum_n \frac{G\mathcal{M}}{R^*}\left(\frac{R^*}{r}\right)^{n+1} \sum_i \sum_m C_{inm} Y_{inm} \qquad (2.2.41a)$$

and

$$\Delta V(r < R^*) = \sum_n \frac{G\mathcal{M}}{R^*}\left(\frac{r}{R^*}\right)^{n} \sum_i \sum_m C_{inm} Y_{inm}. \qquad (2.2.41b)$$

An explicit solution of Dirichlet's problem is obtained through *Poisson's integral*,

$$\Delta V(R > R^*) = \frac{R^*}{4\pi}(r^2 - R^2)\int_S \frac{V(R^*, \phi', \lambda')}{L^3}\,\mathrm{d}S, \qquad (2.2.42)$$

where L is the distance defined by (2.2.6). A similar integral follows for the gravity anomalies (Heiskanen and Moritz 1967)

$$\Delta g(r, \phi, \lambda) = \frac{R^{*2}}{4\pi}\int_S \frac{1}{r}\left(\frac{r^2 - R^{*2}}{L^3} - \frac{1}{r} - \frac{3R^*}{r}\cos\psi\right)\Delta g(R^*, \phi, \lambda)\,\mathrm{d}S. \qquad (2.2.43)$$

SURFACE DENSITY LAYERS

In interpreting the gravity field a question that is often asked is, given a spherical shell of radius r' and thickness Δr within which the density varies laterally, what is the potential or gravity at $r > r'$? In global problems the spherical harmonic approach is again useful. Let the surface density layer be expanded as

$$\mathrm{d}\rho(r', \phi', \lambda') = \sum_i \rho_0(r') \sum_n \sum_m \rho_{inm}(r') Y_{inm}, \qquad (2.2.44)$$

where ρ_{inm} are dimensionless coefficients, then the perturbing potential $\mathrm{P}(r, \phi, \lambda)$ due to this layer

$$\delta V = G\int_S L^{-1} \sum_i \rho_0(r') \sum_n \sum_m \rho_{inm}(r') Y_{inm}(\phi', \lambda')\Delta r\,\mathrm{d}S$$

where L is the distance from $\mathrm{P}(r, \phi, \lambda)$ to the density anomaly at $\mathrm{P}'(r', \phi', \lambda')$. With the expansion (2.2.7) for L^{-1}, the addition theorem (2.2.10), and the orthogonality conditions (2.2.16),

$$\delta V = \frac{4\pi G r'^2}{r} \sum_i \rho_0(r') \sum_n \sum_m \rho_{inm}\, \Delta r \left(\frac{r'}{r}\right)^n \frac{Y_{inm}}{2n+1}. \qquad (2.2.45a)$$

If this is equated with the observed potential (2.2.15c)

$$C_{inm} = \frac{4\pi R^2 \rho_0}{\mathcal{M}(2n+1)} \left(\frac{r'}{R}\right)^{n+2} \rho_{inm}\, \Delta r. \qquad (2.2.46a)$$

The lateral density anomaly may also be interpreted in terms of an irregularly shaped shell

$$r = r'\left\{1 + \sum_i \sum_n \sum_m \mathrm{d}r_{inm} Y_{inm}\right\} \qquad (2.2.44b)$$

across which there is a density contrast $\Delta\rho$. The $\mathrm{d}r_{inm}$ are dimensionless coefficients. Then

$$\Delta V = \frac{4\pi G (r')^3}{r} \sum_i \sum_n \sum_m \Delta\rho\, \mathrm{d}r_{inm} \left(\frac{r'}{r}\right)^n \frac{Y_{inm}}{2n+1} \qquad (2.2.45b)$$

and

$$C_{inm} = \frac{4\pi R^3}{\mathcal{M}(2n+1)} \left(\frac{r'}{R}\right)^{n+3} \Delta\rho\, \mathrm{d}r_{inm}. \qquad (2.2.46b)$$

For density anomalies distributed with depth between radii $r_1 < r' < r_2$ (cf. 2.2.38c)

$$\mathrm{d}\rho(r', \phi', \lambda') = \sum_i {}_k\rho_0 \sum_k \sum_n \sum_m f_k(r)\, {}_k\mathrm{d}\rho_{inm} Y_{inm}$$

$$C_{inm} = \frac{4\pi R^2}{\mathcal{M}(2n+1)} \int_{r_1}^{r_2} f_k(r)\, {}_k\rho_0 \left(\frac{r'}{R}\right)^{n+2} {}_k\mathrm{d}\rho_{inm}\, \mathrm{d}r. \qquad (2.2.46c)$$

2.3. Coordinate reference frames

To define relative positions of points on the Earth's surface and to establish the motion of the planet as a whole in space, it is necessary to introduce two fundamental coordinate reference frames: a *terrestrial* or *Earth-fixed* system $\boldsymbol{x}$ and the *inertial* or *absolute* reference frame $\boldsymbol{X}$. Either system can be defined kinematically by a set of three-dimensional coordinates associated with a polyhedron of points. The axes of the reference frame are then specified by unit vectors ($\hat{\boldsymbol{x}}_i$ or $\hat{\boldsymbol{X}}_i$) equal to $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ obtained from the complete set of coordinates (and their variances) assigned to the points of either system. For the terrestrial frame, the coordinates of the points are tied in some way to the Earth's surface and for the inertial frame the coordinates correspond to stellar sources whose positions are assumed to be fixed in space. An alternative definition of the inertial reference frame is a dynamical one, one in which the equations of motion of a planet or satellite can be adequately defined by the Newtonian equations of motion without requiring the introduction of forces arising from any rotation of the frame.

Different observational methods have been used to determine the coordinates of the polyhedron of the terrestrial stations. Optical observations of astronomical latitudes and longitudes provide the positions of the telescopes in a framework that is defined by the coordinates of the visible stars and by the rotation of the earth. Radio interferometric observations provide the positions in a reference system that is defined by the coordinates of radio sources and the two types of stellar coordinates do not necessarily define the same inertial frame. Observations of the motions of artificial Earth satellites or of the Moon provide a basis for estimating the tracking station positions in a framework that is defined by the satellite's orbital theory and by the planet's rotation. In general, each different type of observation provides its own definition of the reference frame and it will be necessary to specify the relations between them with accuracies that are compatible with both the dictates of geophysics and the technology of the individual measurement process.

Global earth deformations occur at the level of centimetres per year and there is a need to establish the terrestrial network with an internal consistency of a few centimetres. Motions of the Earth as a whole are geophysically significant at the level of about 0.001″ and this is also the accuracy that is approached with modern measurements of the positions of satellites or stellar sources. The deformations of the terrestrial frame produced by the tectonic and geodynamic processes, as well as components of the motion of the Earth's rotation axis, are largely unpredictable and it will be necessary to constantly monitor the relations between the two fundamental reference frames $\boldsymbol{x}$ and $\boldsymbol{X}$.

The establishment of the inertial and terrestrial reference frames involves many detailed and subtle points, some of which are discussed in the volumes edited by Kolaczek and Weiffenbach (1974), Gaposchkin and Kolaczek (1981), Babcock and Wilkins (1988), and Kovalevsky *et al.* (1988) (see also the review by Mueller 1985). With observational precisions now approaching 0.001″ in direction and 1 cm in distance, the definition of accurate inertial reference frames becomes imperative for studies of the Earth's rotation and for global crustal movements and the methodology of establishing such systems remains an active subject for research.

TERRESTRIAL REFERENCE SYSTEMS

The global terrestrial reference system is specified through the Cartesian axes x_i ($i = 1, 2, 3$) that are related to the reference points on the crust in some specified way. For theoretical discussions of the Earth's motion it is sometimes convenient, or merely conventional, to define the frame by the principal axes of the planet, defined such that the moments of inertia I_{ij} ($i \neq j$) (eqn 2.2.23b) vanish. An alternative definition is a set of axes relative to which there are no angular momenta $\boldsymbol{h}$, or

$$\boldsymbol{h} = \int_{\mathcal{M}} \boldsymbol{x} \wedge \boldsymbol{v} \, \mathrm{d}\mathcal{M} = 0 \tag{2.3.1}$$

where the vector $\boldsymbol{v}$ (with components v_i) is the velocity relative to these axes of an element of mass $\mathrm{d}\mathcal{M}$ located at $\boldsymbol{x}$. These are sometimes referred to as *Tisserand's mean axes*. While these definitions are convenient for theoretical developments they are not readily realized because of the movement of mass relative to the crust, in the oceans and atmosphere, and because of the periodic and aperiodic deformations of the solid Earth.

Nearly all methods for determining the global three-dimensional terrestrial coordinates make use of observations of objects outside the Earth; the positions of stars, or the positions of satellites. The planet's rotation axis is therefore an obvious point of reference for defining the frame and a possible definition of one of the axes, x_3, is the mean position of the rotation axis for a specified time interval. A mean position must be used because the rotation axis moves relative to the stations fixed to the crust. Astronomical observations have indicated that this motion is largely unpredictable and includes quasi-periodic amounts of about $0.3''$ amplitude as well as a secular component of about $0.003''\,\mathrm{a}^{-1}$ (per year). This motion of the rotation axis relative to the Earth-fixed reference frame is called *polar motion* (Chapters 5 and 11). It has been observed for the past 80 years by the International Latitude Service and the movements are specified relative to an origin that corresponds to the mean position of the axis for a specified period, usually the years 1900–1905. This is the *Conventional International Origin* (Markowitz 1968). Coordinate systems derived from laser ranging observations, or from long-baseline interferometric observations, use quite different definitions for this origin (see Chapters 6–8). Because the excursions of the instantaneous rotation axis ω from uniform rotation are small it is convenient to define the position of ω relative to x_3 by the two small angles m_1 and m_2 defined in Fig. 2.3. These represent the polar motion or wobble (section 2.5 below).

The x_1–x_2 plane is the equator of x_3 and the origin x_1 can be any arbitrary longitude, such as the mean position of the Greenwich