4

Tides, Rotation, and Shape

What fates impose, that men must needs abide;
It boots not to resist both wind and tide.

William Shakespeare, *Henry VI (3), IV, iii*

4.1 Introduction

So far we have considered all objects as being point masses with no physical dimensions. Since this is evidently not the case for real bodies, we must now consider the effects of the application of universal gravitation to the matter that forms the bodies of the solar system. A *tide* is raised on one body by another because of the effect of the gravitational gradient or the variation of the gravitational force across the body. For example, if we consider the tide raised on a planet by an orbiting satellite, the force experienced by the side of the planet facing the satellite is stronger than that experienced by the far side of the planet. Since none of the bodies that make up the solar system is perfectly rigid, there will be a distortion that gives rise to a *tidal bulge*.

The magnitude of the tidal bulge on a body is determined in part by its internal density distribution and thus, in principle, a measurement of the tidal amplitude could lead to a determination of the internal structure. Such measurements are not possible for any of the planets in the solar system other than the Earth. However, the deforming potential associated with planetary rotation acts in a similar way as that due to tides and a measurement of the rotational deformation of a planet can be used to determine its internal density distribution; this knowledge can then be used to estimate the response of the planet to a tidal potential. Satellites in the solar system that are in synchronous rotation are deformed by both rotational and tidal forces and measurements of their triaxial ellipsoidal figures have been used to determine their internal structures.

The response of the satellite to the tide it has raised can result in dynamical evolution of the system. Since friction is always present to some degree, tides are a dissipative phenomenon and the tide raised by a satellite on a planet can lead to orbital evolution of the satellite and a change in the spin rate of the planet. Just as the satellite raises a tide on the planet, so the planet also raises a tide on the satellite. This can be especially important when the satellite's orbit is eccentric. In some cases the effect of tidal dissipation in a satellite can lead to dramatic consequences, such as the runaway tidal heating of the jovian satellite Io.

4.2 The Tidal Bulge

Consider the case of the tides raised on a planet of mass m_p by a satellite of mass m_s. If we represent the objects as point masses, then Newton's law of gravitation gives the mean magnitude of the mutual force, $\langle F \rangle$, as

$$\langle F \rangle = \mathcal{G}\frac{m_\mathrm{p} m_\mathrm{s}}{r^2}, \tag{4.1}$$

where r is the separation of the centres. If we assume that the bodies move in circular orbits about their common centre of mass (Fig. 4.1), then we have already seen in Sect. 2.7 that the semi-major axes of the orbits are related to the masses by

$$a_\mathrm{s}/a_\mathrm{p} = m_\mathrm{p}/m_\mathrm{s}, \tag{4.2}$$

where the bodies have a constant separation $a = a_\mathrm{p} + a_\mathrm{s}$. The motion of the particle P_1 at the centre of the planet with respect to the centre of mass, C_1, is a circle

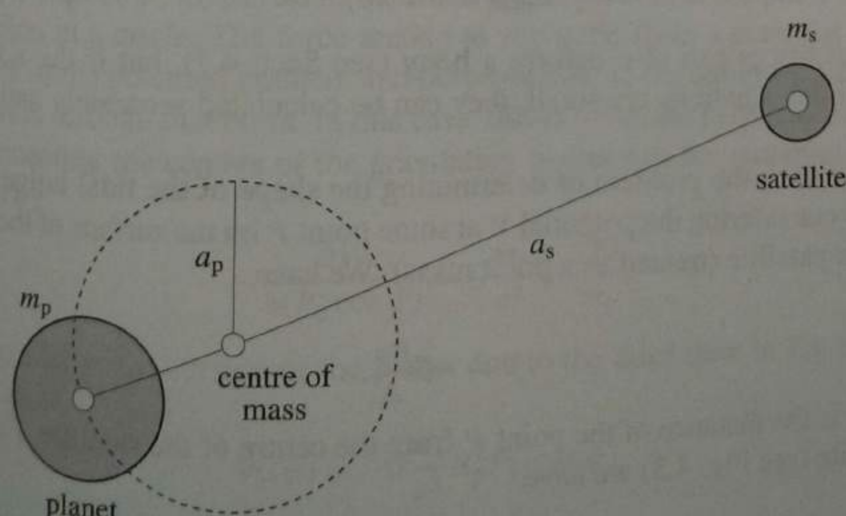

Fig. 4.1. A planet and satellite moving about their common centre of mass in circular orbits. Their semi-major axes with respect to the centre of mass are a_p and a_s, while the semi-major axis of the satellite with respect to the planet is $a = a_\mathrm{p} + a_\mathrm{s}$.

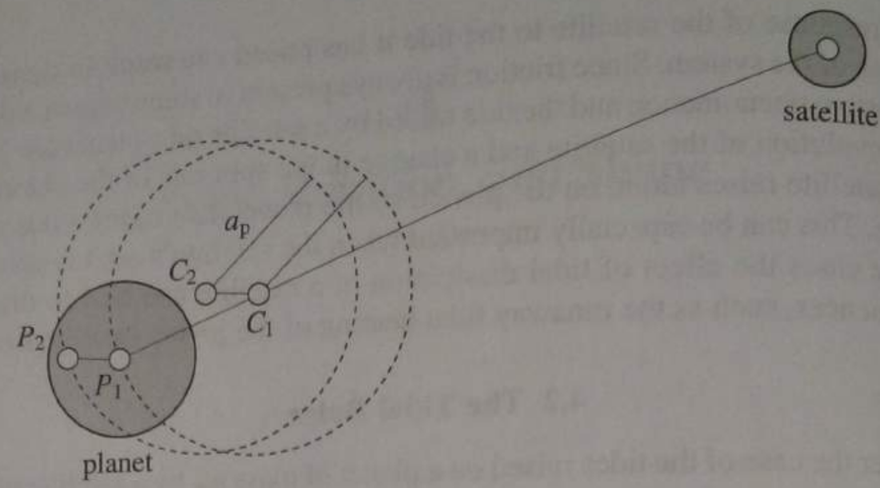

Fig. 4.2. All the particles in the planet move in similar circles of identical radii a_p, but with different centres. The particles P_1 and P_2 are on circles with centres C_1 and C_2, respectively.

of radius a_p. If we neglect rotation it follows that the motion of any other point, P_2, in the planet is a circle of the same radius but with a centre, C_2, displaced from C_1 to the same extent that P_2 is displaced from P_1 (see Fig. 4.2). It follows that all particles that make up the planet are acted on by equal (in magnitude and direction) centrifugal forces but not by equal gravitational forces, **F**. The common centrifugal force is equal to the mean gravitational force, $\langle \mathbf{F} \rangle$, that is,

$$\langle \mathbf{F} \rangle = \text{centrifugal force} \neq \mathbf{F}. \tag{4.3}$$

The tide-generating force, $\mathbf{F}_{\text{tidal}}$, that deforms the planet is defined by

$$\mathbf{F}_{\text{tidal}} = \mathbf{F} - \langle \mathbf{F} \rangle. \tag{4.4}$$

Rotational forces can also deform a body (see Sect. 4.7), but if the tidal and rotational deformations are small, they can be calculated separately and added linearly.

We approach the problem of determining the shape of the tidal bulge on the planet by considering the potential V at some point P on the surface of the planet due to the satellite (treated as a point mass). We have

$$V = -\mathcal{G}\frac{m_s}{\Delta}, \tag{4.5}$$

where Δ is the distance of the point P from the centre of the satellite. From the cosine rule (see Fig. 4.3) we have

$$\Delta = a\left[1 - 2\left(\frac{R_p}{a}\right)\cos\psi + \left(\frac{R_p}{a}\right)^2\right]^{\frac{1}{2}}, \tag{4.6}$$

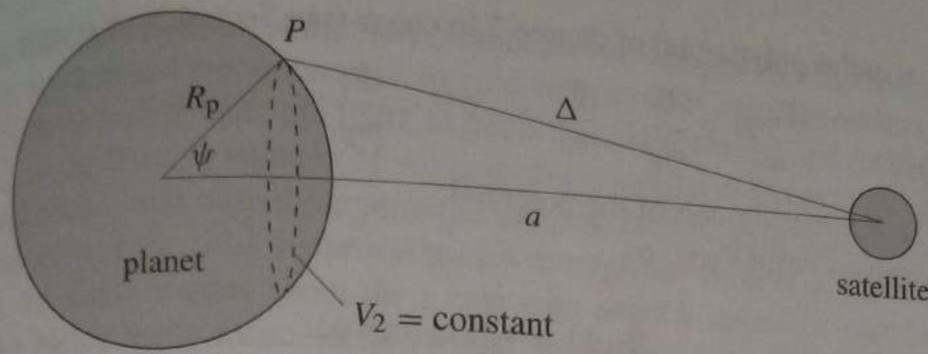

Fig. 4.3. The relationships among the radius of the planet, R_p, the semi-major axis of the satellite, a, and the distance Δ from a point P on the surface of the planet. The dashed line denotes the plane defined by the equipotential surface $V_2 = -\mathcal{G}(m_s/a^2)R_p \cos\psi =$ constant.

where ψ is measured from the line joining the centres of the two bodies. In most cases of interest, $R_p/a \ll 1$. For example, the equatorial radius of the Earth is 6,378 km and the semi-major axis of the Moon's orbit is 384,400 km (see Tables A.4 and A.5). Consequently we can expand Eq. (4.6) binomially to obtain

$$V = -\mathcal{G}\frac{m_s}{a}\left[1 + \left(\frac{R_p}{a}\right)\cos\psi + \left(\frac{R_p}{a}\right)^2 \frac{1}{2}\left(3\cos^2\psi - 1\right) + \cdots\right] \\ \approx V_1 + V_2 + V_3, \tag{4.7}$$

where we have neglected the higher order terms in the expansion.

The first term in Eq. (4.7), $V_1 = -\mathcal{G}m_s/a$, is a constant and since $\mathbf{F}/m_p = -\nabla V$ this term produces no force on the planet. The second term in Eq. (4.7), $V_2 = -\mathcal{G}(m_s/a^2)R_p \cos\psi$, gives rise to the force on the particle at the point P needed for motion in a circle. The force arising at any point from a potential is in the direction of the potential gradient and perpendicular to the equipotential surface that passes through that point. In this case, this is the plane perpendicular to the line connecting the centres of the gravitating bodies and the potential gradient is given by

$$-\frac{\partial V_2}{\partial(R_p \cos\psi)} = \mathcal{G}\frac{m_s}{a^2} = \frac{\langle F\rangle}{m_p}. \tag{4.8}$$

The potential at the surface of the planet due to the third term in Eq. (4.7) can be written as

$$V_3(\psi) = -\mathcal{G}\frac{m_s}{a^3}R_p^2 \mathcal{P}_2(\cos\psi), \tag{4.9}$$

where

$$\mathcal{P}_2(\cos\psi) = \frac{1}{2}\left(3\cos^2\psi - 1\right) \tag{4.10}$$

is the Legendre polynomial of degree 2 in $\cos\psi$ (see Sect. 4.3). Because

$$\frac{\mathbf{F}_{\rm tidal}}{m_{\rm p}} = \frac{\mathbf{F}}{m_{\rm p}} - \frac{\langle\mathbf{F}\rangle}{m_{\rm p}} = -\nabla V - \frac{\langle\mathbf{F}\rangle}{m_{\rm p}} \approx -\nabla V_3(\psi), \tag{4.11}$$

this is the tide raising part of the potential.

We can also write $V_3(\psi)$ as

$$V_3(\psi) = -\zeta g \mathcal{P}_2(\cos\psi), \tag{4.12}$$

where

$$\zeta = \frac{m_{\rm s}}{m_{\rm p}}\left(\frac{R_{\rm p}}{a}\right)^3 R_{\rm p} \tag{4.13}$$

and

$$g = \frac{\mathcal{G}m_{\rm p}}{R_{\rm p}^2} \tag{4.14}$$

is the gravitational acceleration, or surface gravity, of the planet. In this case $\zeta\mathcal{P}_2(\cos\psi)$ is said to be the amplitude of the *equilibrium tide* for any value of ψ on the planet's surface. Note that $\mathcal{P}_2(\cos\psi)$ is a maximum at $\theta = 0$ or π and a minimum at $\theta = \pi/2$ or $3\pi/2$. Given that the Earth rotates on its axis with respect to the stars once every twenty-four hours, this explains why the Moon produces two high tides and two low tides on the Earth approximately every day.

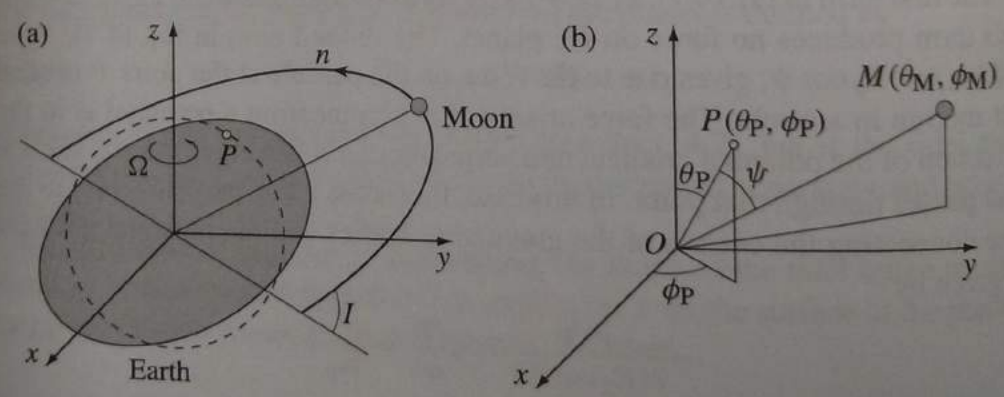

Fig. 4.4. Schematic diagram of the shape of the tidal distortion (solid line) arising from the $V_3(\psi)$ term in the gravitational potential compared with the circular, zero-distortion shape (dashed line). (a) The axis of symmetry of the tidal distortion passes through the centres of Earth and Moon, while the Earth rotates about the z axis with angular speed Ω; the Moon has a mean motion n and an orbital inclination I with respect to the equatorial plane of the Earth. (b) The colatitudes and longitudes of the Moon ($\theta_{\rm M}$, $\phi_{\rm M}$) and a point P ($\theta_{\rm P}$, $\phi_{\rm P}$) on the surface of the Earth. Note that the longitude $\phi_{\rm P}$ (and $\phi_{\rm M}$) is measured from a fixed direction in space and not from a direction that rotates with the Earth.

The tidal deformation of the Earth is made more complicated by the facts that (i) both the Sun and the Moon raise significant tides and (ii) in a geocentric frame both of these bodies orbit the Earth in paths that are eccentric and inclined with respect to the Earth's equator. If we neglect the orbital eccentricities, then the Sun and Moon both raise three principal tides on the Earth. In Fig. 4.4a the axis of symmetry of the tidal distortion passes through the centres of the Earth and Moon, the Earth rotates about the z axis with angular speed Ω, and we allow that the Moon has a mean motion n and an inclination I with respect to the Earth's equatorial plane. In Fig. 4.4b we show the colatitudes and longitudes of the Moon $(\theta_{\rm M}, \phi_{\rm M})$ and of a point P $(\theta_{\rm P}, \phi_{\rm P})$ on the surface of the Earth. Note that the longitudes $\phi_{\rm P}$ and $\phi_{\rm M}$ are measured from a fixed direction in space and not from a direction that rotates with the Earth.

Using the cosine rule we can show that the angle ψ between the position vectors OP and OM is given by

$$\cos\psi = \cos\theta_{\rm P}\cos\theta_{\rm M} + \sin\theta_{\rm P}\sin\theta_{\rm M}\cos(\phi_{\rm P} - \phi_{\rm M}). \tag{4.15}$$

Hence

$$\begin{aligned}\frac{1}{2}\left(3\cos^2\psi - 1\right) = {} & \frac{1}{2}\left(3\cos^2\theta_{\rm P} - 1\right)\frac{1}{2}\left(3\cos^2\theta_{\rm M} - 1\right) \\ & + \frac{3}{4}\sin^2\theta_{\rm P}\sin^2\theta_{\rm M}\cos 2(\phi_{\rm P} - \phi_{\rm M}) \\ & + \frac{3}{4}\sin 2\theta_{\rm P}\sin 2\theta_{\rm M}\cos(\phi_{\rm P} - \phi_{\rm M}). \end{aligned} \tag{4.16}$$

Given that the colatitude $\theta_{\rm P}$ of a fixed point on the Earth is constant, the tidal amplitude at P varies with $\phi_{\rm P}$, $\theta_{\rm M}$, and $\phi_{\rm M}$. The variation of the first term in Eq. (4.16) is determined by the variation with time of $\cos^2\theta_{\rm M} = (1/2)(1 + \cos 2\theta_{\rm M})$. Hence, this term varies with frequency $2n$ and gives rise to a *fortnightly tide*. The second term varies with frequency $2(\Omega - n) \approx 2\Omega$ and gives rise to a *semidiurnal tide*, while the third terms varies with frequency $(\Omega - n) \approx \Omega$ and gives rise to a *diurnal tide*. Because the latter term contains the factor $\sin 2\theta_{\rm M}$, the diurnal tide has a strong fortnightly modulation. Other tidal terms are associated with the Moon's orbital eccentricity.

The corresponding solar *semiannual*, *semidiurnal*, and *diurnal tides* have frequencies $2n_{\rm Sun}$, $2(\Omega - n_{\rm Sun}) \approx 2\Omega$, and $(\Omega - n_{\rm Sun}) \approx \Omega$, where $n_{\rm Sun}$ is the solar (or the Earth's) mean motion. It follows from Eq. (4.13) that the ratio of the amplitudes of the corresponding solar and lunar tides is given, in each case, by

$$\frac{m_{\rm Sun}}{m_{\rm Moon}}\left(\frac{a_{\rm Moon}}{a_{\rm Sun}}\right)^3 \approx 0.46. \tag{4.17}$$

For the tide raised by the Moon on the Earth, $\zeta = 0.36$ m, while for the solar tide $\zeta = 0.16$ m.

4.3 Potential Theory

Before proceeding to the calculation of tidal and rotational deformation it is useful to summarise some results derived from potential theory. The gravitational potential of a homogeneous, spherical body of density γ and radius C can be found by considering the internal and external potentials of a thin spherical shell of radius r, thickness δr, and mass δm (Ramsey 1940). The potential exterior to the shell at some point distant r' from its centre is given by

$$V_{\text{ext}}(r') = -\frac{\mathcal{G}\delta m}{r'} \tag{4.18}$$

and is the same as if all the mass were concentrated at the centre of the shell. Thus the external potential at the surface of a uniform sphere is given by

$$V_{\text{ext}}(C) = -\frac{\mathcal{G}\sum\delta m}{C} = -\frac{4}{3}\pi\gamma\mathcal{G}C^2. \tag{4.19}$$

Because the gravitational force is described by an inverse square law, it follows that the force on a particle interior to the shell is zero. Hence, the gravitational potential interior to the shell must be a constant and can be determined by calculating the potential at any point in the interior. By calculating the potential at the centre of the shell, we obtain

$$V_{\text{int}}(r) = -\frac{\mathcal{G}\sum\delta m}{r} = -4\pi\gamma\mathcal{G}r\,\delta r. \tag{4.20}$$

It follows that the internal potential of a uniform shell of external radius C and internal radius r is given by

$$V_{\text{int}}(C,r) = -4\pi\gamma\mathcal{G}\int_r^C r\,\mathrm{d}r = -2\pi\gamma\mathcal{G}\left(C^2 - r^2\right). \tag{4.21}$$

Hence, the interior ($r < C$) and exterior ($r > C$) potentials of a homogeneous spherical body at some point distant r from its centre are given by

$$V_{\text{int}}(r) = -\frac{4}{3}\pi\gamma\mathcal{G}r^2 - 2\pi\gamma\mathcal{G}\left(C^2 - r^2\right) = -\frac{2}{3}\pi\gamma\mathcal{G}\left(3C^2 - r^2\right), \tag{4.22}$$

$$V_{\text{ext}}(r) = -\frac{4}{3}\pi\gamma\mathcal{G}\frac{C^3}{r}. \tag{4.23}$$

The internal and external potentials of a deformed body can be expressed in terms of spherical harmonic functions. The following brief discussion of the use of these functions in potential theory is based on the fuller accounts given by Ramsey (1940), MacRobert (1967), Bullen (1975), and Blakely (1995).

The gravitational potential V in free space satisfies Laplace's equation:

$$\nabla^2 V = 0. \tag{4.24}$$

A function V is said to be *homogeneous of degree n* if it satisfies *Euler's equation*:

$$x\frac{\partial V}{\partial x} + y\frac{\partial V}{\partial y} + z\frac{\partial V}{\partial z} = nV. \tag{4.25}$$

Homogeneous functions that also satisfy Laplace's equations are called *spherical solid harmonics*. They have the important property that when transformed into spherical coordinates, they can be factored into three functions, each of which depends on only one of the three variables r, θ, or ϕ. (For a good account of spherical harmonic analysis, see Blakely (1995).)

Using spherical polar coordinates (r, θ, ϕ), where r is the radial distance from the centre of mass, θ is the colatitude measured from the polar axis, and ϕ is a longitude measured from some arbitrary fixed direction, we can write Laplace's equation as

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)+\frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial V}{\partial \mu}\right)+\frac{1}{(1-\mu^2)}\frac{\partial^2 V}{\partial \phi^2}=0, \tag{4.26}$$

where $\mu = \cos\theta$. Laplace's equation can be solved by substituting the trial solution $V = r^n S_n(\mu, \phi)$, where $S_n(\mu, \phi)$ is independent of r, into Eq. (4.26). This gives

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right)=n(n+1)r^n S_n, \tag{4.27}$$

and Eq. (4.26) reduces to

$$\frac{\partial}{\partial \mu}\left((1-\mu^2)\frac{\partial S_n}{\partial \mu}\right)+\frac{1}{1-\mu^2}\frac{\partial^2 S_n}{\partial \phi^2}+n(n+1)S_n=0. \tag{4.28}$$

This is called *Legendre's equation* and any function S_n that satisfies this equation is called a *spherical surface harmonic* of degree n. Because $n(n+1)$ remains unchanged when we write $-(n+1)$ for n, the general solution of Laplace's equation can be written as

$$V=\sum_{n=0}^{\infty}\left(A_n r^n + B_n r^{-(n+1)}\right)S_n(\mu,\phi). \tag{4.29}$$

Each of the terms in this equation is called a *solid harmonic* of degree n and $-(n+1)$, respectively (Ramsey 1940).

In the applications discussed in this chapter, where the deformations, either tidal or rotational, have axial symmetry, the solution of Legendre's equation reduces to

$$(1-\mu^2)\frac{\partial^2\mathcal{P}_n(\mu)}{\partial \mu^2}-2\mu\frac{\partial \mathcal{P}_n(\mu)}{\partial \mu}+n(n+1)\mathcal{P}_n(\mu)=0, \tag{4.30}$$

where Legendre's polynomial, $\mathcal{P}_n(\mu)$, is given by the terminating series

$$\mathcal{P}_n(\mu)=\frac{1\cdot 3\cdot 5\ldots(2n-1)}{n!}\left[\mu^n-\frac{n(n-1)}{2(2n-1)}\mu^{n-2}+\frac{n(n-1)(n-2)(n-3)}{2\cdot 4\cdot(2n-1)(2n-3)}\mu^{n-4}+\cdots\right] \tag{4.31}$$

or by Rodrigues's formula

$$\mathcal{P}_n(\mu) = \frac{1}{2^n n!}\frac{\mathrm{d}^n(\mu^2-1)^n}{\mathrm{d}\mu^n} \tag{4.32}$$

and the colatitude θ is measured from the axis of symmetry of the deformation. Legendre polynomials are *zonal harmonics*; the first five are given by

$$\mathcal{P}_0(\mu) = 1, \tag{4.33}$$

$$\mathcal{P}_1(\mu) = \mu = \cos\theta, \tag{4.34}$$

$$\mathcal{P}_2(\mu) = \frac{1}{2}(3\mu^2-1) = \frac{1}{4}(3\cos 2\theta + 1), \tag{4.35}$$

$$\mathcal{P}_3(\mu) = \frac{1}{2}(5\mu^3-3\mu) = \frac{1}{8}(5\cos 3\theta + 3\cos\theta), \tag{4.36}$$

$$\mathcal{P}_4(\mu) = \frac{1}{8}(35\mu^4-30\mu^2+3) = \frac{1}{64}(35\cos 4\theta + 20\cos 2\theta + 9)\,. \tag{4.37}$$

Surface harmonics are orthogonal functions and double integrals of their products over the surface of the sphere have the following useful properties. The element of area on a unit sphere ($r = 1$) is given by $\sin\theta\,\mathrm{d}\theta\,\mathrm{d}\phi = -\mathrm{d}\mu\,\mathrm{d}\phi$. If $Y_m(\mu,\phi)$ and $S_n(\mu,\phi)$ are two surface harmonics of degrees m and n, respectively, and $m \neq n$, then

$$\int_0^{2\pi}\int_{-1}^{+1} Y_m(\mu,\phi)\,S_n(\mu,\phi)\,\mathrm{d}\mu\,\mathrm{d}\phi = 0\,. \tag{4.38}$$

If the two surface harmonics have the same degree, n, and one is a zonal harmonic, $\mathcal{P}_n(\mu)$, then

$$\int_0^{2\pi}\int_{-1}^{+1} S_n(\mu,\phi)\,\mathcal{P}_n(\mu)\,\mathrm{d}\mu\,\mathrm{d}\phi = \frac{4\pi}{2n+1}S_n(1), \tag{4.39}$$

where $S_n(1)$ is the value of $S_n(\mu,\phi)$ at the pole of $\mathcal{P}_n(\mu)$.

Consider two points on a unit sphere: a variable point (θ',ϕ') and a fixed point (θ,ϕ). Let ψ be the angle subtended by these points at the centre of the sphere, and consider the integral

$$\int_0^{2\pi}\int_{-1}^{+1} S_n(\theta',\phi')\,\mathcal{P}_n(\cos\psi)\,\mathrm{d}\mu'\,\mathrm{d}\phi'\,. \tag{4.40}$$

Let the axes of the coordinates of the variable point be changed such that the new axis defining the variable colatitude passes through the fixed point (θ,ϕ), and let the new angular coordinates of the variable point be (Θ',Φ'), such that $\Theta' = \psi$. Also let $S_n(\theta',\phi')$ become $Y_n(\Theta',\Phi')$. Then, from Eq. (4.39), we obtain

$$\int_0^{2\pi}\int_{-1}^{+1} Y_n(\Theta',\Phi')\,\mathcal{P}_n(\cos\Theta')\,\mathrm{d}(\cos\Theta')\,\mathrm{d}\Phi' = \frac{4\pi}{2n+1}Y_n(1)\,. \tag{4.41}$$

However,

$$Y_n(1) = S_n(\theta,\phi)\,. \tag{4.42}$$

It follows from this that

$$\int_0^{2\pi}\int_{-1}^{+1} S_n(\mu',\phi')\,\mathcal{P}_n(\cos\psi)\,\mathrm{d}\mu'\,\mathrm{d}\phi' = \frac{4\pi}{2n+1}S_n(\mu,\phi), \tag{4.43}$$

where $S_n(\mu,\phi)$ is the same function of (μ,ϕ) that $S_n(\mu',\phi')$ is of (μ',ϕ') (MacRobert 1967).

Now consider the potential at some fixed point P due to a homogeneous, nearly spherical body whose surface is defined by

$$R(\theta') = C\left[1+\epsilon_2\mathcal{P}_2(\cos\theta')\right], \tag{4.44}$$

where $\epsilon_2(\ll 1)$ is a constant and C is now the mean radius. The point P can be either internal ($r < C$) or external ($r > C$) and has spherical coordinates (r,μ,ϕ), where $\mu = \cos\theta$ and the colatitude θ is measured from the axis of symmetry of the tidal bulge (Fig. 4.5). The total gravitational potential at P is the sum of two parts. The part due to the spherical body is given by Eqs. (4.22) and (4.23), while the other, noncentral, part of the potential is due to the thin distribution of matter between the surface of the deformed body and the mean sphere. At some point $P'(r',\mu',\phi')$, the radial thickness of this thin layer of matter is $\epsilon_2 C\mathcal{P}_2(\mu')$ and the element of volume at that point is $\epsilon_2 C^3\mathcal{P}_2(\mu')\,\mathrm{d}\mu'\,\mathrm{d}\phi'$. The potential at P due to the element of mass is determined by the distance $PP' = \Delta$, where

$$\frac{1}{\Delta} = \left(C^2 + r^2 - 2Cr\cos\psi\right)^{-1/2}. \tag{4.45}$$

If $r < C$, then

$$\frac{1}{\Delta} = \frac{1}{C}\left[1+\left(\frac{r}{C}\right)^2 - 2\mu\left(\frac{r}{C}\right)\right]^{-1/2}, \tag{4.46}$$

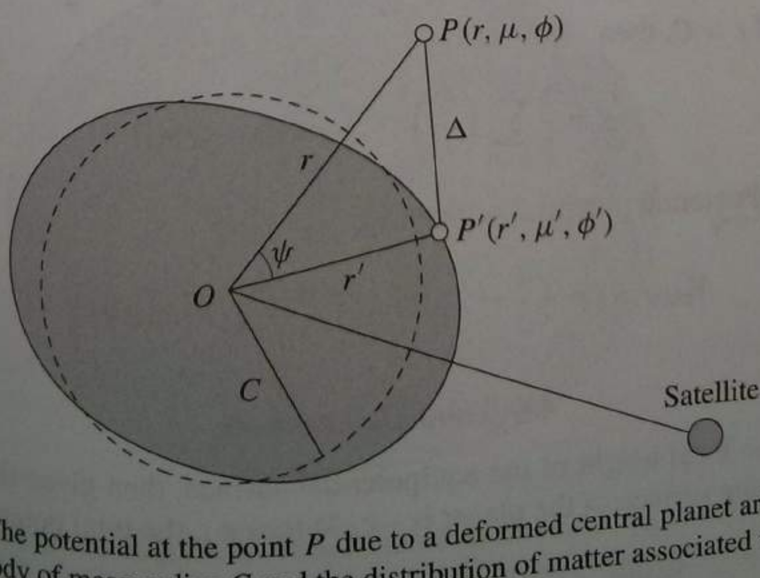

Fig. 4.5. The potential at the point P due to a deformed central planet arises from the spherical body of mean radius C and the distribution of matter associated with the tidal bulge.

and on expanding binomially and ordering the terms in powers of r/C we obtain

$$\frac{1}{\Delta} = \frac{1}{C}\left[1 + \left(\frac{r}{C}\right)\mu + \left(\frac{r}{C}\right)^2\left(-\frac{1}{2} + \frac{3}{2}\mu^2\right) + \left(\frac{r}{C}\right)^3\left(-\frac{3}{2}\mu + \frac{5}{2}\mu^3\right) + \cdots\right], \tag{4.47}$$

and inspection of Eqs. (4.33)–(4.37) shows that Eq. (4.47) can be written as

$$\frac{1}{\Delta} = \frac{1}{C}\sum_{n=0}^{\infty}\left(\frac{r}{C}\right)^n \mathcal{P}_n(\cos\psi) + \mathcal{O}(\epsilon_2). \tag{4.48}$$

Hence, the noncentral contribution to the total potential at P is given by

$$V_{\text{nc,int}} = -\gamma\mathcal{G}C^2\epsilon_2\sum_{n=0}^{\infty}\left(\frac{r}{C}\right)^2\int\int\mathcal{P}_2(\mu')\mathcal{P}_n(\cos\psi)\,\mathrm{d}\mu'\,\mathrm{d}\phi'. \tag{4.49}$$

From Eq. (4.43) we obtain

$$\sum_{n=0}^{\infty}\left(\frac{r}{C}\right)^n\int\int\mathcal{P}_2(\mu')\mathcal{P}_n(\cos\psi)\,\mathrm{d}\mu'\,\mathrm{d}\phi' = \frac{4\pi}{5}\left(\frac{r}{C}\right)^2\mathcal{P}_2(\cos\theta). \tag{4.50}$$

Hence, the internal, noncentral contribution to the potential at P is given by

$$V_{\text{nc,int}} = -\frac{4\pi}{5}\gamma\mathcal{G}r^2\epsilon_2\mathcal{P}_2(\cos\theta), \tag{4.51}$$

and the total internal potential at P is the sum of this term and that given by Eq. (4.22), that is,

$$V_{\text{int}}(r,\theta) = -\frac{4}{3}\pi C^3\gamma\mathcal{G}\left[\frac{3C^2 - r^2}{2C^3} + \frac{3}{5}\frac{r^2}{C^3}\epsilon_2\mathcal{P}_2(\cos\theta)\right]. \tag{4.52}$$

Similarly, if $r > C$, then

$$\frac{1}{\Delta} = \frac{1}{r}\sum_{n=0}^{\infty}\left(\frac{C}{r}\right)^n \mathcal{P}_n(\cos\psi) + \mathcal{O}(\epsilon_2) \tag{4.53}$$

and the total external potential is given by

$$V_{\text{ext}}(r,\theta) = -\frac{4}{3}\pi C^3\gamma\mathcal{G}\left[\frac{1}{r} + \frac{3}{5}\frac{C^2}{r^3}\epsilon_2\mathcal{P}_2(\cos\theta)\right]. \tag{4.54}$$

4.4 Tidal Deformation

If $h(\psi)$ is the local height of the equipotential surface, then given that the tidal potential at the surface of the planet is $-\zeta g\mathcal{P}_2(\cos\psi)$, the total potential is

$$V_{\text{total}}(r,\psi) = -\frac{\mathcal{G}m_{\text{p}}}{B} + gh(\psi) - \zeta g\mathcal{P}_2(\cos\psi), \tag{4.55}$$

where B is the mean radius of the planet. On an equipotential surface this must be independent of ψ; hence we must have $h(\psi) = \zeta \mathcal{P}_2(\cos\psi)$, where it is understood that ψ is measured from the axis of symmetry of the tidal bulge. The equilibrium tide defines the shape of a shallow, zero-density ocean covering an inflexible, spherical planet. In practice, of course, no fluid has zero density and no solid is inflexible. We need to find the tidal deformation of a real solid or fluid body.

In order to show the factors involved, we consider the case of a simple, two-component model planet that has a mean radius B, a homogeneous, incompressible, fluid ocean of density σ, and a homogeneous, incompressible, solid core of mean radius A, density ρ, and rigidity μ (Street 1925, Dermott 1979a; see Fig. 4.6). The rigidity μ is a measure of the force needed to deform the elastic body.

The equilibrium tide is the second-order surface harmonic that describes the equipotential surface that would exist close to a completely rigid, ocean-free, spherical planet circled by a single, distant satellite. If the planet is not completely rigid and has an ocean, then the surfaces of the ocean and the equilibrium tide will *not* coincide. This is the case even if we neglect effects due to the kinetic energy of the ocean currents. To calculate the response of the ocean and core to the gravitational field of the satellite we must take into account the effect of the gravitational field of the tidal bulge itself (self-gravitation) and the effect of the elastic forces within the solid. We must also allow for the elastic yielding of the core under the action of all the forces that act on both the core and the ocean; that is, we must allow for the potentials of the deformed core and the ocean tide

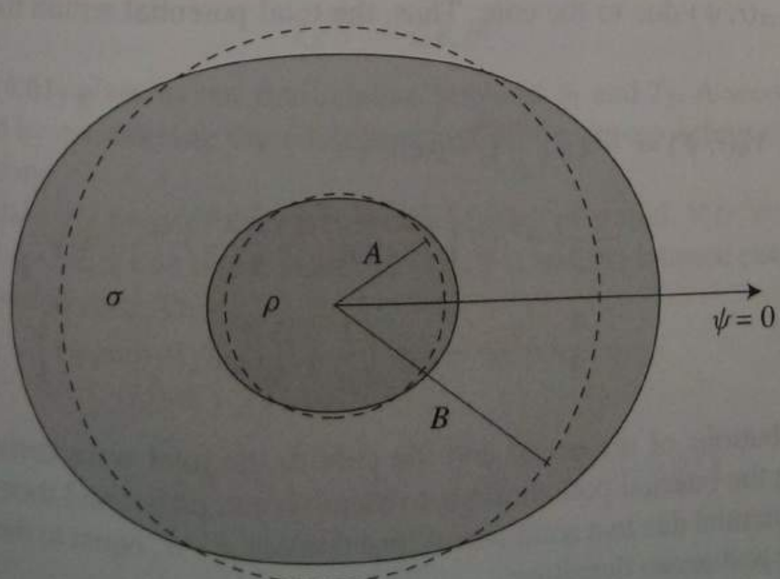

Fig. 4.6. Schematic diagram showing a model planet consisting of a deformed core of mean radius A and density ρ, surrounded by a deformed ocean of mean radius B and density σ. The circles of radii A and B are denoted by dashed lines.

itself, as well as the loading effects of the tides raised in the various parts of the planet (Street 1925).

Since the tide-raising potential is a solid spherical harmonic of the second degree, it follows that the deformation of the planet must be described in terms of the same harmonic function (Love 1911). If this were not the case, then it would not be possible for the surface of the ocean, for example, to be an equipotential surface. We have already seen in Sect. 4.2 that the tidal potential depends only on one angle, ψ, implying axial symmetry about the line joining the two centres. Thus, we can describe the deformed shapes of the core boundary and ocean surface by

$$R_{cb}(\psi) = A\,[1 + S_2\mathcal{P}_2(\cos\psi)] \tag{4.56}$$

and

$$R_{os}(\psi) = B\,[1 + T_2\mathcal{P}_2(\cos\psi)] \tag{4.57}$$

respectively, where S_2 and T_2 are constants. We will determine S_2 and T_2 from (i) the fact that the surface of the static ocean must be an equipotential and (ii) by considering the equilibrium of all the forces acting on the mean core boundary.

The potential within the ocean, $V_o(r, \psi)$, is the sum of three potentials: (i) the tidal potential due to the satellite,

$$V_3(r,\psi) = -\frac{\mathcal{G}m_s}{a^3}r^2\mathcal{P}_2(\cos\psi) = -\zeta g\left(\frac{r}{B}\right)^2\mathcal{P}_2(\cos\psi), \tag{4.58}$$

which is a generalisation of Eqs. (4.9) and (4.12), (ii) $V_{int}(r, \psi)$ due to the ocean, and (iii) $V_{ext}(r, \psi)$ due to the core. Thus, the total potential within the ocean is given by

$$\begin{aligned} V_o(r,\psi) = &-\zeta g\left(\frac{r}{B}\right)^2\mathcal{P}_2(\cos\psi) \\ &-\frac{4}{3}\pi B^3\sigma\mathcal{G}\left(\frac{3B^2-r^2}{2B^3}+\frac{3}{5}\frac{r^2}{B^3}T_2\mathcal{P}_2(\cos\psi)\right) \\ &-\frac{4}{3}\pi A^3(\rho-\sigma)\mathcal{G}\left(\frac{1}{r}+\frac{3}{5}\frac{A^2}{r^3}S_2\mathcal{P}_2(\cos\psi)\right). \end{aligned} \tag{4.59}$$

The contributions of the ocean and the core to the total were derived by first calculating the internal potential due to an all-ocean planet and then adding the external potential due to a solid core with a density, $\rho - \sigma$, equal to the difference of the core and ocean densities.

We calculate the potential at the ocean surface, $V_{os}(r, \psi)$, by substituting the surface equation, $r = B\,[1 + T_2\mathcal{P}_2(\cos\psi)]$, into Eq. (4.59) and expanding the resulting expression, neglecting second and higher order terms in ζ/B, S_2 and

T_2. This gives

$$V_{os}(r,\psi) = -\zeta g \mathcal{P}_2(\cos\psi) - \frac{4}{3}\pi B^2 \sigma \mathcal{G}\left(\frac{1}{2} - \frac{2}{5}T_2\mathcal{P}_2(\cos\psi)\right) - \frac{4}{3}\pi\frac{A^3}{B}(\rho-\sigma)\mathcal{G}\left(1 - T_2\mathcal{P}_2(\cos\psi) + \frac{3}{5}\left(\frac{A}{B}\right)^2 S_2\mathcal{P}_2(\cos\psi)\right). \quad (4.60)$$

The terms in this equation that do not depend on ψ give rise to compressive forces only; since we have assumed that the core and ocean are incompressible, these have no role in determining the shape of the planet and can be ignored. Given that the ocean surface is an equipotential, the sum of those terms that do depend on ψ must be zero. Hence we must have

$$\frac{\zeta_c}{A} = \left[\frac{2}{5}\frac{\sigma}{\rho} + \left(\frac{A}{B}\right)^3\left(1 - \frac{\sigma}{\rho}\right)\right]T_2 - \frac{3}{5}\left(\frac{A}{B}\right)^5\left(1 - \frac{\sigma}{\rho}\right)S_2. \quad (4.61)$$

Here we have introduced

$$\zeta_c = \frac{m_s}{m_c}\left(\frac{A}{a}\right)^3 A, \quad (4.62)$$

where m_c is the mass of the core and

$$g_c = \frac{\mathcal{G}m_c}{A^2} \quad (4.63)$$

is the gravity at the core boundary. The quantity ζ_c is the amplitude of the "equilibrium tide" that would exist at the core boundary if the ocean were removed; it is related to ζ in Eq. (4.13) by

$$\zeta g A^2 = \zeta_c g_c B^2. \quad (4.64)$$

Equation (4.61) gives us our first relation between S_2 and T_2. A second relation is obtained by considering the equilibrium of all the forces acting on the mean core boundary.

Within the solid body of the core, the deforming potential, $V_c(r,\psi)$ is the sum of the tidal potential due to the satellite, $V_3(r,\psi)$, and the internal potentials due to the ocean and core. Thus,

$$V_c(r,\psi) = -\zeta_c g_c\left(\frac{r}{A}\right)^2\mathcal{P}_2(\cos\psi) - \frac{4}{3}\pi B^3\sigma\mathcal{G}\left(\frac{3B^2-r^2}{2B^3} + \frac{3}{5}\frac{r^2}{B^3}T_2\mathcal{P}_2(\cos\psi)\right) - \frac{4}{3}\pi A^3(\rho-\sigma)\mathcal{G}\left(\frac{3A^2-r^2}{2A^3} + \frac{3}{5}\frac{r^2}{A^3}S_2\mathcal{P}_2(\cos\psi)\right). \quad (4.65)$$

Consider a sphere within the core, concentric with the planet's centre. For a fixed value of r, the effective deforming potential is given by those terms in $V_c(r,\psi)$

that only depend on $\mathcal{P}_2(\cos\psi)$. The other terms are compressive and can be ignored. Hence the effective deforming potential can be written as

$$V_c(r,\psi) = -Zr^2\mathcal{P}_2(\cos\psi), \tag{4.66}$$

where

$$Z = \frac{g_c}{A}\left(\frac{\zeta_c}{A} + \frac{3}{5}\frac{\sigma}{\rho}(T_2 - S_2) + \frac{3}{5}S_2\right). \tag{4.67}$$

Chree (1896a) showed that the yielding of the core under the force resulting from this deforming potential is the same as that which would be produced by an outward normal force per unit area of amount $\rho ZA^2\mathcal{P}_2(\cos\psi)$ acting at the mean core boundary, $r = A$.

Other pressures also act at this boundary. These arise from loading terms caused by the hydrostatic pressure in the ocean and the tide in the solid core. For example, in the case of a shallow ocean for which $(B - A) \ll B$ and there is no variation of g within the ocean, these pressures arise from (i) the variation with ψ of the ocean depth and (ii) the variation with ψ of the distance of the core boundary from the centre of the planet. These pressures are given by the product of the local gravity, the density, and the height of the tide and can be written as

$$P_o(\psi) = g\sigma B(T_2 - S_2)\mathcal{P}_2(\cos\psi) \tag{4.68}$$

and

$$P_c(\psi) = g_c\rho AS_2\mathcal{P}_2(\cos\psi). \tag{4.69}$$

For a deep ocean g is not a constant and although the loading term due to the tide in the core remains the same, the angular variation of the hydrostatic pressure on the core is no longer determined by the height of the ocean tide alone. We must also take account of the angular variation of gravity within the ocean.

In the general case of a deep ocean, the pressure on the core boundary due to the ocean is given by

$$P_o(\psi) = \int_{R_{cb}}^{R_{os}} \sigma(r)\frac{\partial V_o(r,\psi)}{\partial r}\mathrm{d}r, \tag{4.70}$$

where the integral limits are from the core boundary to the ocean surface. As the ocean in our model is assumed to be incompressible and of uniform density, this reduces to

$$P_o(\psi) = \sigma\left[V_o(R_{os},\psi) - V_o(R_{cb},\psi)\right]. \tag{4.71}$$

Furthermore, since only the variable part of the potential can contribute to the deforming forces (we are neglecting compression) and since the surface of the ocean is an equipotential surface, it follows that $V_o(R_{os},\psi)$ is a constant and can be neglected.

We can obtain the potential at the core boundary, $V_{cb}(\psi)$, by substituting the equation for the core boundary, $R_{cb} = A[1 + S_2\mathcal{P}_2(\cos\psi)]$, into Eq. (4.60) or

Eq. (4.65) and expanding the resulting expression, neglecting second and higher order terms in ζ/B, S_2, and T_2. This gives

$$V_{cb}(\psi) = \text{constant} - Ag_c\left(\frac{\zeta_c}{A} + \frac{3}{5}\frac{\sigma}{\rho}(T_2 - S_2) - \frac{2}{5}S_2\right)\mathcal{P}_2(\cos\psi), \qquad (4.72)$$

where the ψ dependence is contained in the second term on the right-hand side of the equation; this is the part that will contribute to the variable part of the pressure at the core boundary, $P_o(\psi)_\psi$. Hence

$$P_o(\psi)_\psi = \sigma Ag_c\left(\frac{\zeta_c}{A} + \frac{3}{5}\frac{\sigma}{\rho}(T_2 - S_2) - \frac{2}{5}S_2\right)\mathcal{P}_2(\cos\psi). \qquad (4.73)$$

The total effective outward normal force per unit area at the mean core boundary is the sum of the elastic forces within the core and the loading terms due to the ocean and the core tides. We write this as $X\mathcal{P}_2(\cos\psi)$, where

$$\begin{aligned} X &= \rho A^2 Z - P_o(\psi)_\psi - \rho g_c A S_2 \\ &= \frac{2}{5}\rho g_c A\left(1 - \frac{\sigma}{\rho}\right)\left(\frac{5}{2}\frac{\zeta_c}{A} - S_2 + \frac{3}{2}\frac{\sigma}{\rho}(T_2 - S_2)\right). \end{aligned} \qquad (4.74)$$

Note that $X \to 0$ as $\sigma \to \rho$, which must be the case if the ocean is in hydrostatic equilibrium.

Love (1944) showed that the radial displacement of the solid core produced by this deforming pressure is

$$\Delta R(\psi) = \frac{5}{19}\frac{A}{\mu}X\mathcal{P}_2(\cos\psi), \qquad (4.75)$$

which must be the same as $AS_2\mathcal{P}_2(\cos\psi)$. This gives us our second relationship between S_2 and T_2:

$$S_2 = \frac{1}{\tilde{\mu}}\left(1 - \frac{\sigma}{\rho}\right)\left(\frac{5}{2}\frac{\zeta_c}{A} - S_2 + \frac{3}{2}\frac{\sigma}{\rho}(T_2 - S_2)\right), \qquad (4.76)$$

where $\tilde{\mu}$, the *effective rigidity* of the solid core, is a dimensionless quantity defined by

$$\tilde{\mu} = \frac{19\mu}{2\rho g_c A}. \qquad (4.77)$$

It is a measure of the ratio of the elastic and gravitational forces acting at the core boundary.

If $\tilde{\mu} \ll 1$, then the core responds like a fluid, whereas for $\tilde{\mu} \gg 1$ elastic forces within the core dominate. Note that if $\sigma = \rho$, then $S_2 = 0$ and the elastic core is undeformed. If the planet is ocean free, then $\sigma = 0$ and, from Eq. (4.76), the amplitude of the tide on the isolated core is given by

$$AS_2 = \frac{(5/2)\zeta_c}{1 + \tilde{\mu}}. \qquad (4.78)$$

In the general case, we can write

$$AS_2 = F\frac{(5/2)\zeta_c}{1+\tilde{\mu}} \quad \text{and} \quad BT_2 = H\frac{5}{2}\zeta, \tag{4.79}$$

in which case F is a dimensionless quantity that is a measure of the effect of the ocean on the amplitude of the core tide, and H is a measure of the effect of the internal structure on the external shape of the planet. By eliminating T_2 from Eqs. (4.61) and (4.76), we obtain

$$F = \frac{(1+\tilde{\mu})(1-\sigma/\rho)(1+3/2\alpha)}{1+\tilde{\mu}-\sigma/\rho+(3\sigma/2\rho)(1-\sigma/\rho)-(9/4\alpha)(A/B)^5(1-\sigma/\rho)^2} \tag{4.80}$$

and

$$H = \frac{2\langle\rho\rangle}{5\rho}\left(\frac{1+\tilde{\mu}+(3/2)(A/B)^2F\delta}{(1+\tilde{\mu})(\delta+2\sigma/5\rho)}\right), \tag{4.81}$$

where

$$\alpha = 1+\frac{5}{2}\frac{\rho}{\sigma}\left(\frac{A}{B}\right)^3\left(1-\frac{\sigma}{\rho}\right) \quad \text{and} \quad \delta = \left(\frac{A}{B}\right)^3\left(1-\frac{\sigma}{\rho}\right) \tag{4.82}$$

and $\langle\rho\rangle$ is the mean density. If the planet is fluid throughout, or if because of thermal creep the solid core has relaxed to hydrostatic equilibrium (this has application to tidal bulges on satellites with spin rates equal to or *synchronous* with their orbital mean motions), then $\tilde{\mu} = 0$ and the hydrostatic value of H is given by

$$H_h = \frac{2\langle\rho\rangle}{5\sigma}\left(\frac{1+(3\delta/5\gamma)(A/B)^2}{\delta+2\sigma/5\rho-(9\delta\sigma/25\gamma\rho)(A/B)^2}\right), \tag{4.83}$$

where

$$\gamma = \frac{2}{5}+\frac{3\sigma}{5\rho} \tag{4.84}$$

(Dermott 1979a).

We can now use these results to determine the tidal amplitudes in the limiting case of a planet with a shallow, uniform ocean. If $A = B$, $\zeta_c = \zeta$, and $\langle\rho\rangle = \rho$ the condition for the ocean surface to be an equipotential surface, Eq. (4.61), reduces to

$$\frac{\zeta}{A} = \frac{2}{5}S_2+\left(1-\frac{3}{5}\frac{\sigma}{\rho}\right)(T_2-S_2)\,. \tag{4.85}$$

Eliminating the terms containing S_2 alone from Eqs. (4.61) and (4.76) we obtain

$$A(T_2-S_2) = \frac{\zeta\tilde{\mu}}{1-\sigma/\rho+\tilde{\mu}\,(1-3\sigma/5\rho)} \tag{4.86}$$

for the amplitude of the ocean tide. This agrees with that first given by Chree (1896b). We note that $A(T_2 - S_2) \to 0$ as $\tilde{\mu} \to 0$. Furthermore, if $\sigma = \rho$, then

$$A(T_2 - S_2) = \frac{5}{2}\zeta \tag{4.87}$$

and is independent of $\tilde{\mu}$. Thus, when the core and ocean have the same density, the core is undeformed and the amplitude of the ocean tide is a factor 5/2 larger than the "equilibrium" tide given by Eq. (4.13).

We can find the amplitude of the solid body tide in the case of a shallow ocean by substituting the expression for $T_2 - S_2$ from Eq. (4.86) into Eq. (4.61). This gives

$$AS_2 = \frac{5}{2}\zeta \left[\frac{(1 - \sigma/\rho)}{1 - \sigma/\rho + \tilde{\mu}(1 - 3\sigma/5\rho)} \right]. \tag{4.88}$$

For an ocean-free planet, $\sigma = 0$ and

$$AS_2 = \frac{(5/2)\zeta}{1 + \tilde{\mu}}, \tag{4.89}$$

which agrees with the expression first given by Lord Kelvin (Thompson 1863). Kelvin applied this result to observations of the fortnightly tide for which $AS_2 \approx 0.6\zeta$ and hence deduced that the rigidity of the Earth is $\sim 1.2 \times 10^{11}\,\mathrm{Nm}^{-2}$ or $\sim 50\%$ greater than the rigidity of uncompressed steel. At the time, this was a surprising result because the interior of the Earth was then thought to be largely molten (Bullen 1975).

In the case of the Earth it is perhaps more realistic to consider the planet as being composed of a solid core of density ρ and rigidity μ covered by a shallow ocean of density σ. In which case, the amplitudes of the tides in the ocean and the solid core are given by Eqs. (4.86) and (4.88). However, these amplitudes were calculated by assuming that the ocean is in hydrostatic equilibrium and thus that the tidal currents in the ocean have no role in determining the shape of the ocean surface. For the Earth, this is not a good assumption.

The effects of ocean currents depend on the natural oscillation frequency of the ocean basin and this is determined by the size, shape, and depth of the basin. For heuristic purposes we follow Proudman (1953) and consider a uniform, equatorial canal of depth d bounded by two parallels of latitude. In a reference frame centred on the Earth and rotating with the mean motion of the Moon the tidal bulge is stationary. Hence the speed of the bulge with respect to the surface of the Earth is $U = 2\pi A/T_E \approx 500\,\mathrm{ms}^{-1}$, where T_E is the spin period of the Earth. However, U is not the speed of the tidal current in the ocean. The fluid in the tidal bulge is supplied by a current of speed u flowing uniformly throughout the depth of the ocean (see Fig. 4.7). If, at a given longitude, ζ denotes the height of the surface above the ocean floor, then from the equation of continuity we

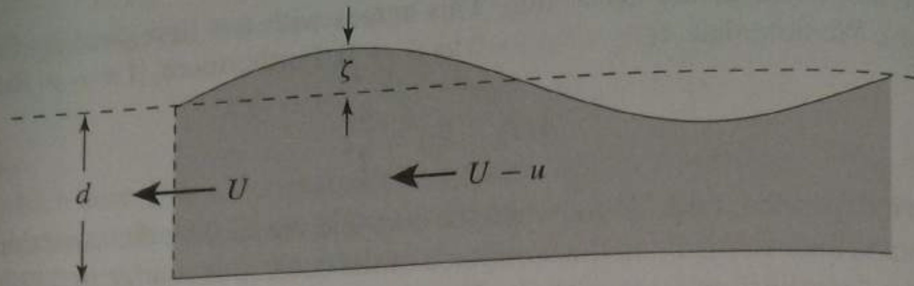

Fig. 4.7. Schematic diagram of a tidal wave in a uniform equatorial canal of depth d bounded by two parallels of latitude. The tidal bulge is stationary in a reference frame centred on the Earth and rotating with the Moon. The speed of the bulge with respect to the solid Earth is U. However, the fluid in the tidal bulge is supplied by a current of speed u, which, at a given longitude, flows uniformly throughout the depth of the ocean.

have

$$(U - u)(d + \zeta) = Ud\,. \tag{4.90}$$

If $\zeta \ll d$, then

$$u = \frac{\zeta}{d}U\,. \tag{4.91}$$

The mean ocean depth on the Earth is $\sim$ 4 km, and so $u \sim 0.1\ \mathrm{ms}^{-1}$. From Bernouilli's theorem, which relates work done by the hydrostatic pressure to the kinetic and potential energies of the fluid in streamline flow, we obtain

$$\frac{1}{2}(U - u)^2 + g\zeta + \Psi = \text{ constant}, \tag{4.92}$$

where Ψ is the tidal potential. If we assume that at the surface of the canal $\Psi = -g\bar{\zeta}$, then given that U^2 is a constant and $u^2 \ll uU$, we have

$$Uu = g(\zeta - \bar{\zeta})\,. \tag{4.93}$$

Substituting for u from Eq. (4.91) we obtain

$$\zeta = \frac{\bar{\zeta}}{1 - U^2/gd}\,. \tag{4.94}$$

For the Earth resonance would occur in an equatorial canal of depth $d_{\mathrm{res}} = U^2/g \approx 22$ km and given that the mean depth of the ocean is less than d_{res} we would expect the tides to be inverted. This reasoning cannot be applied to the Earth as a whole because for near-global oceans we cannot assume that the total tidal potential is decoupled from the tidal currents. However, Eq. (4.94) provides a good indication that in calculating the shapes of the Earth's oceans we must include effects due to the ocean currents and consider the possibility of resonance in the various ocean basins.

4.5 Rotational Deformation

In Sect. 4.4 we showed how the tide raised by an orbiting satellite distorts the surface of a planet. By modelling the planet as consisting of a core and mantle we were able to derive expressions for the distortions of each component. The most important result is that the shape of the distorted planet (see Fig. 4.6) can be approximated by an *oblate spheroid* with long semi-axis a lying along the planet–satellite line and with short semi-axes $b = c$ giving a circular cross section perpendicular to the axis of symmetry (the planet–satellite line). The spheroid is modelled using $\mathcal{P}_2(\cos\psi)$, a Legendre polynomial of degree 2 where the angle ψ is measured from the axis of symmetry. In this section we show that many of the analytical results derived for tidal distortion can be directly applied to rotational distortion.

Consider a spherical, rigid planet rotating at an angular rate Ω (see Fig. 4.8). A point P on the surface experiences a centrifugal acceleration, $\mathbf{a}_{\mathrm{cf},x} = \Omega^2 r \sin\theta\,\hat{\mathbf{x}}$, or, since $x = r\sin\theta$, $\mathbf{a}_{\mathrm{cf},x} = \Omega^2 x\,\hat{\mathbf{x}}$. By symmetry a similar point located on the surface in the y–z plane feels an acceleration $\mathbf{a}_{\mathrm{cf},y} = \Omega^2 y\,\hat{\mathbf{y}}$. The rotation of the planet produces no acceleration along the axis of rotation. Therefore, an arbitrary point on the surface at position (x, y, z) experiences an acceleration

$$\mathbf{a}_{\mathrm{cf}} = \Omega^2 (x\,\hat{\mathbf{x}}, y\,\hat{\mathbf{y}})\,. \tag{4.95}$$

We can consider this centrifugal acceleration in terms of a centrifugal potential, V_{cf}, such that $\mathbf{a}_{\mathrm{cf}} = -\nabla V_{\mathrm{cf}}$, where, in polar coordinates,

$$V_{\mathrm{cf}}(r,\theta) = -\frac{1}{2}\Omega^2 r^2 \sin^2\theta\,. \tag{4.96}$$

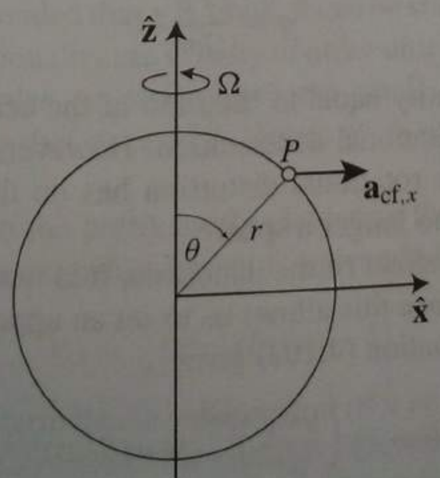

Fig. 4.8. The acceleration experienced at a point P (in the x–z plane) on the surface of a planet rotating at a rate Ω. Here θ is the angle measure from the z axis (the axis of rotation) and r is the radial distance to the point.

Now consider a global ocean on the surface of the planet. The fluid experiences a total potential

$$V_{\mathrm{total}}(r,\theta) = -\frac{\mathcal{G}m_{\mathrm{p}}}{r} + V_{\mathrm{cf}}(r,\theta)\,. \tag{4.97}$$

We know that, in equilibrium, the surface of the fluid must lie on an equipotential surface where the surface is locally perpendicular to the net gravitational and centrifugal acceleration. Assuming that the distortion of the ocean surface from a sphere is small, we can write

$$r_{\mathrm{ocean}} = a + \delta r(\theta), \tag{4.98}$$

where $a = r_{\mathrm{equatorial}}$, the equatorial radius of the planet. Therefore the potential describing the surface is the constant given by

$$V_{\mathrm{total}}(\mathrm{surface}) \approx -\frac{\mathcal{G}m_{\mathrm{p}}}{a} + \frac{\mathcal{G}m_{\mathrm{p}}}{a^2}\delta r - \frac{1}{2}\Omega^2 a^2 \sin^2\theta - \Omega^2 a \sin^2\theta\,\delta r\,. \tag{4.99}$$

For most planets $\Omega^2 a \ll \mathcal{G}m_{\mathrm{p}}/a^2$ (see below) and hence we can neglect the last term in Eq. (4.99). Hence

$$\delta r \approx \mathrm{constant} + \frac{\Omega^2 a^4}{2\mathcal{G}m_{\mathrm{p}}}\sin^2\theta\,. \tag{4.100}$$

Therefore the effect of rotation, as might be expected, is to cause flattening at the poles. We can quantify the extent of the deformation by defining the *oblateness* or *flattening* of the planet as

$$f = \frac{r_{\mathrm{equatorial}} - r_{\mathrm{pole}}}{r_{\mathrm{equatorial}}}\,. \tag{4.101}$$

The analysis above suggests that for planets we should expect to find $f \approx q/2$, where

$$q = \frac{\Omega^2 a^3}{\mathcal{G}m_{\mathrm{p}}} \tag{4.102}$$

is a dimensionless quantity equal to the ratio of the centrifugal acceleration at the equator to the gravitational acceleration. However, we have neglected the feedback effect that the rotational distortion has on the planet's gravity field given that the planet is no longer a sphere.

Before including the effect of the distortion, it is worthwhile to consider the extreme case, $q \to 1$, since this allows us to set an upper limit on the rotational velocity of a planet. Equation (4.102) gives

$$\Omega_{\mathrm{max}} \approx \left(\frac{\mathcal{G}m_{\mathrm{p}}}{a^3}\right)^{1/2} \approx a\,(\mathcal{G}\langle\rho\rangle)^{1/2}\,, \tag{4.103}$$

where $\langle\rho\rangle$ is the mean density of the planet. For the Earth, $\langle\rho\rangle = 5.52\ \mathrm{g\ cm^{-3}}$, and so $\Omega_{\mathrm{max}} \approx 1.2\times10^{-3}\ \mathrm{rad\ s^{-1}}$ for a rotation period of $P_{\mathrm{min}} = 1.4$ h. For Jupiter, $P_{\mathrm{min}} \approx 2.9$ h compared with its current value of 9.9 h.

Because of rotational flattening, most planets (but *not* most satellites) may be treated to a good approximation as oblate spheroids (i.e., triaxial ellipsoids with two equal long axes ($a = b$) and one short axis (c)). A basic result of potential theory is that the *external* gravitational potential of any body with an axis of symmetry can be written in the form

$$V_{\text{gravity}}(r, \theta) = -\frac{\mathcal{G}m}{r}\left[1 - \sum_{n=2}^{\infty} J_n \left(\frac{R}{r}\right)^n \mathcal{P}_2(\cos\theta)\right], \qquad (4.104)$$

where m is the total mass, R (= a for the case of rotational deformation) is the equatorial radius, J_n are dimensionless constants, and, as before (see Sect. 4.3), $\mathcal{P}_n(\cos\theta)$ are Legendre polynomials of degree n. Note that there is no $n = 1$ term provided the origin of the coordinates is chosen as the body's centre of mass. The J_ns reflect the distribution of mass within the body and must be determined empirically for a planet. Of these quantities, by far the most important is J_2 and this has a simple physical interpretation in terms of the three moments of inertia, $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$, about the principal axes. MacCullagh's theorem (see Eq. (5.36) and the derivation by Cook 1973) allows us to write (Cook 1980)

$$J_2 = \frac{\mathcal{C} - \frac{1}{2}(\mathcal{A} + \mathcal{B})}{ma^2} \approx \frac{\mathcal{C} - \mathcal{A}}{ma^2}, \qquad (4.105)$$

where the approximation is valid when $\mathcal{A} \approx \mathcal{B}$, as is the case for rotational distortion. In general, J_n is given by the integral

$$J_n = +\frac{1}{mR^n}\int_0^R \int_{-1}^{+1} r^n \mathcal{P}_n(\mu)\rho(r, \mu)\, 2\pi r^2\, \mathrm{d}\mu\, \mathrm{d}r, \qquad (4.106)$$

where $\mu = \cos\theta$ and $\rho(r, \mu)$ is the internal density distribution. Since $\mathcal{P}_2(\mu)$ is an odd function for odd n, $J_3 = J_5 = J_7 = \cdots = 0$ for a planet whose northern and southern hemispheres are symmetric. In fact, only the Earth has a measured nonzero value of J_3. Provided that q is small, it can be shown that $J_n \propto q^{n/2}$ where the constants of proportionality are usually of order unity. Therefore, since $q \ll 1$ generally, the higher-order J_ns rapidly become small. For a planet of uniform density it can be shown that $J_2 = q/2$. Values of J_2 and J_4 for the planets are listed in Table A.4.

We can now return to the problem of calculating the flattening of a rotating planet. We can write the centrifugal potential given in Eq. (4.96) as

$$V_{\text{cf}} = \frac{1}{3}\Omega^2 r^2 \left[\mathcal{P}_2(\mu) - 1\right]. \qquad (4.107)$$

Here we can ignore the term that is independent of μ so that V_{cf} has the same angular dependence as the J_2 term in the planetary gravity field. The total potential experienced by an ocean on the surface of the planet is now

$$V_{\text{total}}(r, \theta) = -\frac{\mathcal{G}m_{\text{p}}}{r} + \left[\frac{\mathcal{G}m_{\text{p}}a^2}{r^3} J_2 + \frac{1}{3}\Omega^2 r^2\right]\mathcal{P}_2(\mu), \qquad (4.108)$$

where we have neglected J_4, J_6, etc. As before, we require the surface to be an equipotential and write $r = R + \delta r(\theta)$. On substituting into Eq. (4.108) and expanding this gives

$$\delta r = \text{constant} - \left[J_2 + \frac{1}{3} q \right] R \mathcal{P}_2(\mu) . \qquad (4.109)$$

We can use the definition of f in Eq. (4.101) with this new expression for δr to obtain

$$f = \frac{3}{2} J_2 + \frac{1}{2} q , \qquad (4.110)$$

replacing our previous result, $f \approx q/2$. Using data from Yoder (1995) we can compare the calculated value of f with its observed value. For the Earth $f_{\mathrm{calc}} = 0.003349$ whereas $f_{\mathrm{obs}} = 0.003353$. In the case of Jupiter $f_{\mathrm{calc}} = 0.06670$ whereas $f_{\mathrm{obs}} = 0.06487$. This value is sufficiently large that it is possible to see polar flattening in images of Jupiter's disc. From these comparisons it is clear that Eq. (4.110) provides a good estimate of the flattening.

The fact that both tidal deformation and rotational deformation give rise to a surface that can be modelled by means of a Legendre polynomial of degree two implies that the theory developed in Sect. 4.4 for the tidal deformation of a planet with a core and mantle can be directly applied to the case of rotational deformation. In both cases measurements of the extent of the deformation reveal information about the internal structure of the planet. Of course, the theory is equally applicable to the case of a satellite deformed by (i) the tides raised on it by the planet and (ii) the satellite's own rotation. In Sect. 4.7 we examine the particular case of the deformations on a satellite in synchronous rotation.

The J_2 of a planet modifies the gravitational field experienced by an orbiting object such as a satellite or ring particle. The main consequence is that the elliptical path of an orbiting object rotates or *precesses* in space. The dynamical consequences are discussed in more detail in Sects. 6.11, 7.7, 7.9, and 8.11. For our purposes it is sufficient to know that the precessional effect of J_2 can be observed directly by monitoring the orbits of satellites and narrow, eccentric rings. Therefore, J_2 is an observable quantity and Eq. (4.105) allows us to relate it to $C - \mathcal{A}$, the difference in the two principal moments of inertia.

However, we still require an additional relation between C and $\mathcal{A}$ in order to calculate each separately and thereby constrain models of the interior. One such method for the case of the Earth is to observe the consequences of the torque exerted by the Sun and Moon on the rotationally flattened Earth. This causes the Earth's spin axis to rotate about the normal to Earth's orbit plane at a rate that is proportional to $(C - \mathcal{A})/C$ (Cook 1980), an effect called *luni-solar precession*. At the moment this technique for determining C and $\mathcal{A}$ can only be applied to the Earth–Moon system. For the other planets a different method must be employed.

4.6 The Darwin–Radau Relation

The Darwin–Radau relation (see, for example, Cook 1980) is an approximate equation relating the moment of inertia factor, C/mR^2 (where m is the mass of the object and R is its mean radius), to the values of q, f, and J_2 of the planet or satellite. The relation was first derived by Radau (1885) based on work by Clairaut (1743); Darwin (1899) also contributed to the problem. The underlying assumption is that the object concerned is in hydrostatic equilibrium. The relation can be expressed in a number of different forms but here we adopt the one used by Cook (1980). This gives

$$\frac{C}{mR^2} = \frac{2}{3}\left[1 - \frac{2}{5}\left(\frac{5}{2}\frac{q}{f} - 1\right)^{1/2}\right]. \quad (4.112)$$

If we define the moment of inertia factor, $\bar{C}$, a dimensionless quantity, to be

$$\bar{C} = \frac{C}{mR^2} \quad (4.113)$$

and make use of the relationship between J_2, q, and f given in Eq. (4.110) then we can rewrite the Darwin–Radau relation given in Eq. (4.112) as

$$\frac{J_2}{f} = -\frac{3}{10} + \frac{5}{2}\bar{C} - \frac{15}{8}\bar{C}^2. \quad (4.114)$$

However, the relationship between $\bar{C}$ and J_2/f given by this form of the Darwin–Radau relation is just the limiting case of a more general result that can be derived using the distortion model for a more realistic planet derived in Sect. 4.4. For the core–mantle model derived there, Dermott (1979b) gives

$$\bar{C} = \frac{2}{5}\left[\frac{\sigma}{\langle\rho\rangle} + \left(1 - \frac{\sigma}{\langle\rho\rangle}\right)\left(\frac{A}{B}\right)^2\right]. \quad (4.115)$$

This result can be derived from first principles using the definition of the moment of inertia and the known dimensions and distortions of the core and mantle. If the satellite surface and the core–mantle interface are in equilibrium then the factor H has the hydrostatic value H_h given by Eq. (4.83). Dermott (1979b) relates the value of J_2/f to H_h by means of the equation

$$\frac{J_2}{f} = \frac{2}{3}\left(1 - \frac{2}{5H_h}\right). \quad (4.116)$$

The formulae for δ and γ allow us to relate J_2/f to $\bar{C}$ for a given value of A/B. We obtain

$$\frac{J_2}{f} = \frac{2}{3} + \frac{\bar{C} - \frac{2}{5}(A/B)^2}{1 - (A/B)^2} + \frac{8 - 20(A/B)^5 + 10\bar{C}\left[5(A/B)^3 - 2\right]}{12\left[(A/B)^5 - 1\right] + 15\bar{C}\left[2 - 5(A/B)^3 + 3(A/B)^5\right]}. \quad (4.117)$$

There are two limiting cases of this formula to consider:

- In the case of a point core, $A/B \to 0$ and

$$J_2/f \to \bar{C}. \tag{4.118}$$

- In the case of the Darwin–Radau relation, $A/B \to 1$ and

$$J_2/f \to -\frac{3}{10} + \frac{5}{2}\bar{C} - \frac{15}{8}\bar{C}^2. \tag{4.119}$$

Figure 4.9 shows plots of J_2/f as a function of $\bar{C}$ for the point core model, the Darwin–Radau model, and a general core–mantle model with $A/B = 0.5$. The known values of J_2/f for Earth and the giant planets are indicated with horizontal lines denoting the limits on $\bar{C}$ for each planet. Note that the range of values of $\bar{C}$ for a given value of J_2/f becomes smaller as J_2/f increases. The key point is that measurements of J_2/f allow limits to be placed on the moment of inertia factor. When this information is combined with measurements of $C - A$ from Eq. (4.105) we can obtain estimates for A as well as C. Such estimates provide constraints on detailed models of planetary interiors.

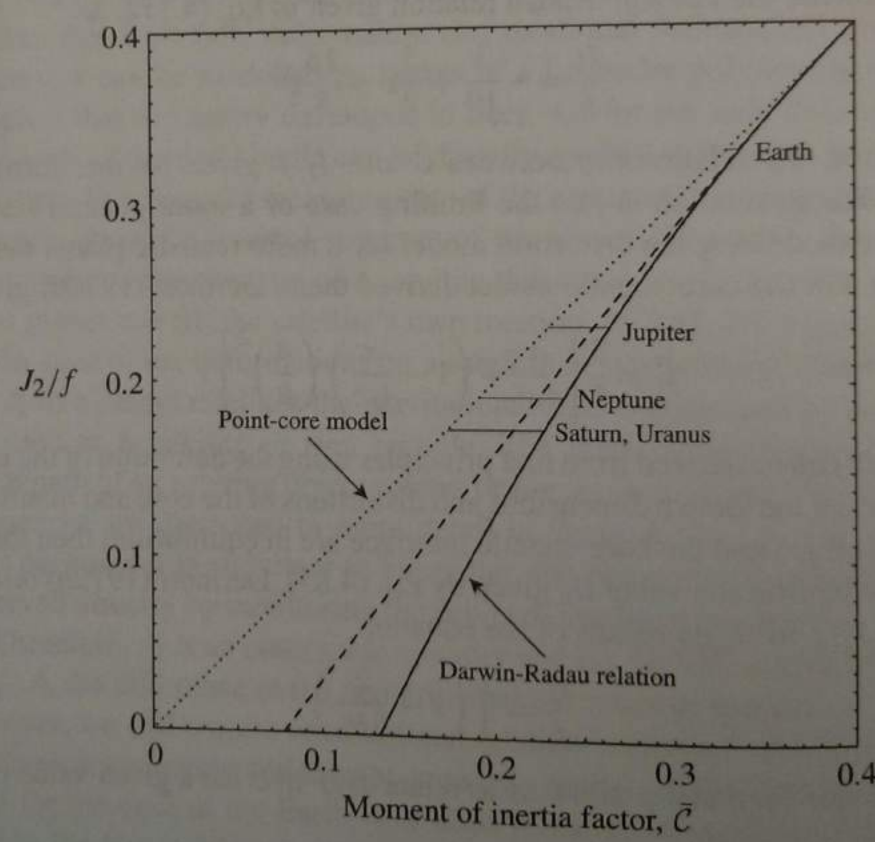

Fig. 4.9. The value of J_2/f as a function of the moment of inertia factor, $\bar{C}$, for the cases of (i) a point-core model (dotted line), (ii) a model with $A/B = 0.5$ (dashed line), and (iii) the Darwin–Radau relation (solid line). Values of J_2/f for the Earth and the giant planets are indicated.

4.7 Shapes and Internal Structures of Satellites

Consider the case of a satellite in hydrostatic equilibrium. The satellite is assumed to exhibit synchronous rotation and to be moving in a near-equatorial, near-circular orbit about a planet. The satellite experiences tidal deformation due to the planet as well as rotational deformation due to its own spin. The theory developed above and the fact that the satellite's mean motion n is equal to its spin frequency Ω allows us to show that the resulting shape of the satellite should be a triaxial ellipsoid. Indeed, because there are specific relationships between the semiaxes for a satellite in hydrostatic equilibrium, accurate measurements of a satellite's shape allow us to determine if it is in equilibrium and when combined with other data allow us to infer properties of the internal structure of the satellite.

The centrifugal potential at a point (r, θ, ψ) arising from the satellite's rotation is

$$V_{\text{rotational}} = \frac{1}{3}\Omega^2 r^2 \mathcal{P}_2(\cos\theta) \tag{4.120}$$

(cf. Eq. (4.107)), where θ is the angle between the radius vector and the vertical axis, and $\mathcal{P}_2(\cos\theta) = (1/4)(3\cos 2\theta + 1)$ is the Legendre polynomial of degree two. Note that $V_{\text{rotational}}$ is independent of ψ, the angle between the projected radius vector and the x–y plane; this accounts for the symmetry of the equipotential surface about the z axis. The tidal potential due to the planet at the same point is

$$V_{\text{tidal}} = -\frac{\mathcal{G}m_{\text{p}}}{a^3} r^2 \mathcal{P}_2(\cos\psi) \tag{4.121}$$

(cf. Eq. (4.9)), where m_{p} is the mass of the planet. Note that V_{tidal} is independent of θ, the angle the radius vector makes with the z axis; this results in the symmetry of the equipotential surface about the x axis. However, by making use of Kepler's third law and noting that $n = \Omega$ we can write

$$V_{\text{tidal}} = -\Omega^2 r^2 \mathcal{P}_2(\cos\psi), \tag{4.122}$$

and hence $V_{\text{rotational}}$ has exactly the same form as V_{tidal}, differing only by a factor 3 in magnitude and with different axes of symmetry. This means that we can apply the theory developed in Sect. 4.4 for tidal deformation directly to rotational deformation. Figure 4.10 shows a comparison of the resulting equipotential surfaces for each type of deformation. Note that in the case of rotational deformation (Fig. 4.10a) the shape is symmetric with respect to the z axis whereas in the case of tidal deformation the x axis (lying along the satellite–planet line) is the axis of symmetry.

Using the shape model for the surface of the mantle (i.e., the surface of the satellite) given by Eq. (4.57) we can now calculate the resulting shape for each form of deformation, recognising that an addition factor of $-1/3$ has to be

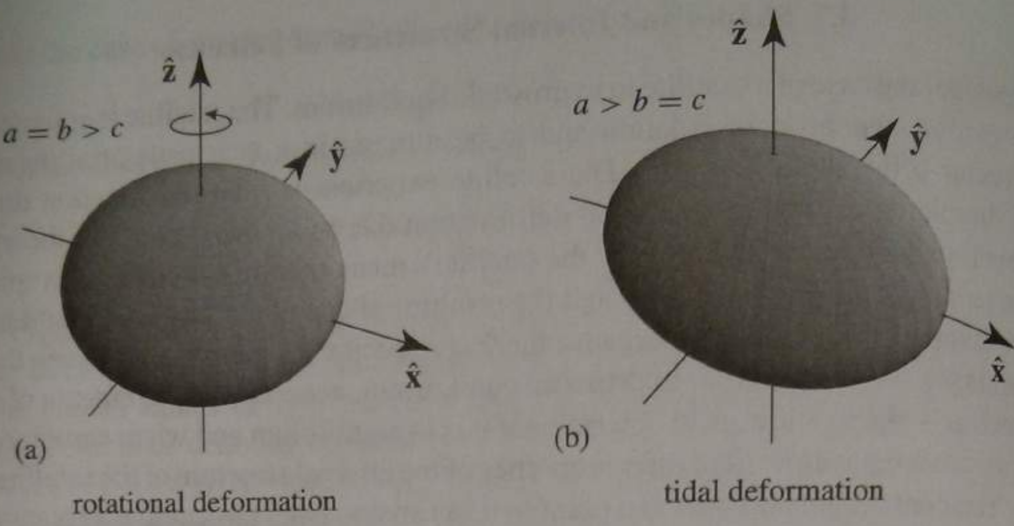

Fig. 4.10. Examples of the equipotential surfaces arising from (a) rotational deformation, where the z axis is the axis of rotation, and (b) tidal deformation, where the tide-raising body lies along the direction of the x axis.

introduced for the rotational case. It is easiest to define the shape in terms of the semiaxes a, b, and c (along the x, y, and z axes) of a general, triaxial ellipsoid. Each of these quantities can be calculated as a function of B and T_2 alone by evaluating the Legendre polynomial for the appropriate values of θ and ψ.

For the rotational deformation we need only calculate $\mathcal{P}_2(\cos\theta)$ at $\theta = \pi/2$ to give a and b, and at $\theta = 0$ to give c. This gives

$$a_r = B(1 + T_2/6), \qquad b_r = B(1 + T_2/6), \qquad c_r = B(1 - T_2/3)\,. \tag{4.123}$$

For the tidal deformation we need only calculate $\mathcal{P}_2(\cos\psi)$ at $\psi = 0$ to give a and at $\psi = \pi/2$ to give b and c. This gives

$$a_t = B(1 + T_2), \qquad b_t = B(1 - T_2/2), \qquad c_t = B(1 - T_2/2)\,. \tag{4.124}$$

If we assume that the spin axis is perpendicular to the orbit plane and that we can add the rotational and tidal contributions linearly, the resulting shape is a triaxial ellipsoid with semiaxes

$$a = B(1 + 7T_2/6), \qquad b = B(1 - T_2/3), \qquad c = B(1 - 5T_2/6)\,. \tag{4.125}$$

In particular, we have the result that the shape of a synchronously rotating satellite in hydrostatic equilibrium subjected to rotational and tidal deformations is such that

$$b - c = \frac{1}{4}(a - c) \tag{4.126}$$

(Dermott 1979b). Furthermore, adapting Eq. (4.13) for the case of the tide raised by the planet on the satellite and combining it with Eq. (4.102) gives $\zeta/B = 3q/4$.

Hence, using the definition of BT_2 given in Eq. (4.79),

$$a - c = 2BT_2 = \frac{15}{4} H_h q B \, . \tag{4.127}$$

In this equation the quantities $a - c$, B, and q can be measured from sufficiently high-resolution images of the satellite. Combining these with knowledge of the mass of the satellite gives its mean density, $\langle \rho \rangle$. From the definition of the factor H_h in Eq. (4.83) we see that this now places constraints on the values of A/B (where A is the mean radius of the core) as well as σ and ρ (the densities of the core and mantle, respectively). This is the basis of a technique that can be used to determine the internal structure of satellites, particularly those close to a planet where the tidal and rotational distortions are large.

Dermott (1979b) used the theory presented here and in Sects. 4.5 and 4.6 to suggest that spacecraft determinations of (a) the gravitational moments of satellites such as Io, Ganymede, and Titan and (b) the shapes of satellites such as Mimas and Tethys could be used to provide evidence for internal differentiation. Dermott & Thomas (1988) applied a second-order version of the shape technique to Mimas using high-resolution images obtained by the *Voyager* spacecraft. They found that the shape of Mimas is a good approximation to a triaxial ellipsoid with measurements giving $(b-c)/(a-c) = 0.27 \pm 0.04$ compared with a predicted ratio of 0.25 from Eq. (4.126); this suggests that the satellite is close to hydrostatic equilibrium. Dermott & Thomas combined their estimated value of the mean radius, $B = 198.8$ km, with the mass of Mimas as determined by Kozai (1957) to obtain a mean density $\langle \rho \rangle = 1.137 \pm 0.018$ g cm^{-3}. They showed that $a - c = 16.9 \pm 0.7$ km compared with a predicted value of 20.3 ± 0.3 km for an undifferentiated satellite. The lower than expected bulge is strongly suggestive of a centrally condensed satellite. One model of its interior that is consistent with these observations predicts a rocky core of dimensions $A/B = 0.44 \pm 0.09$ with an ice mantle of density $\rho = 0.96 \pm 0.08$ g cm^{-3}. Another possibility is that Mimas has a deep regolith composed of porous (and hence significantly underdense) ice. It is intriguing to note that observations of the dynamical interaction of Janus and Epimetheus, the coorbital satellites of Saturn (see Sect. 3.12), suggest an icy composition of comparable porosity.

The orbital tour of the *Galileo* spacecraft around the jovian system has brought it to within 1,000 km or less of the surfaces of all four of the Galilean satellites. Using spacecraft tracking data Anderson et al. (1996a, 1996b, 1997a) have derived estimates of the moment of inertia factor for Io ($\bar{C} = 0.378 \pm 0.007$), Europa ($\bar{C} = 0.347 \pm 0.014$), Ganymede ($\bar{C} = 0.311 \pm 0.003$), and Callisto ($\bar{C} = 0.406 \pm 0.030$). Recall that if a satellite in hydrostatic equilibrium is homogeneous then we would expect to find $\bar{C} = 0.4$. Therefore, Io, Europa, and Ganymede are all centrally condensed. Indeed, Ganymede has the lowest measured value of $\bar{C}$ of any object in the solar system. However, the initial data

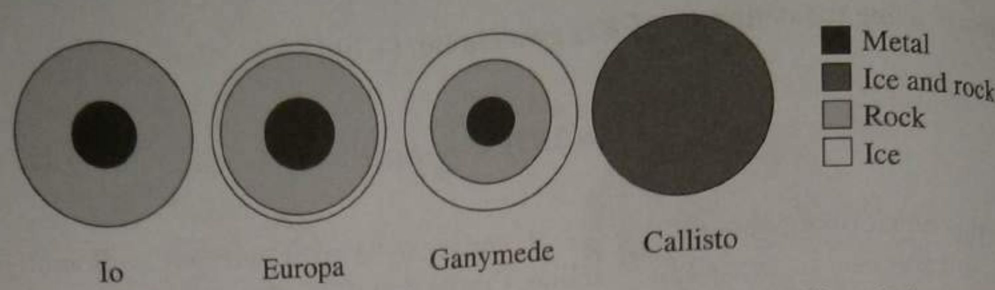

Fig. 4.11. Plausible models for the interiors of the Galilean satellites derived from spacecraft data. All the satellites are drawn to a common radial scale. The data are taken from Anderson et al. (1996a, 1996b, 1997a, 1997b).

for Callisto (Anderson et al. 1997b) suggested that it could be undifferentiated, with further encounters hinting at a partial separation of rock and ice (Anderson et al. 1998). Combined with data on the satellites' mean densities, it has been possible to derive models for the interiors. These are shown schematically in Fig. 4.11. The tidal heating of Io is discussed in Sect. 4.11, but here we note that one of the more intriguing possibilities arising from interpretation of the *Galileo* data is that the tidal heating of Europa has resulted in an ocean of liquid water below a crust of water ice.

Although we have assumed that the satellite is in a near-circular orbit, in reality the shape of a satellite changes with the varying tidal potential as the satellite moves around an orbit that is appreciably eccentric. In circumstances where a spacecraft has repeated, close approaches with a satellite, and measurements of $\bar{C}$ and J_2 can be derived at each encounter, it is possible to get extremely accurate measurements of the moments, as well as information about such physical properties as the satellite's rigidity. The *Cassini* spacecraft's use of repeated gravity assists from the saturnian satellite Titan should provide unprecedented information about its internal structure (Rappaport et al. 1997).

4.8 The Roche Zone

Consider a small, spherical satellite of mass m_s and radius R_s in synchronous rotation about a planet of mass m_p and radius R_p. The semi-major axis of the satellite's circular orbit is taken to be a and its mean motion is n. We have already seen in Sect. 3.6 that the unstable Lagrangian equilibrium points L_1 and L_2 lie on the line connecting the planet and satellite centres at a distance d_L from the satellite, where

$$d_L = \left(\frac{m_s}{3m_p}\right)^{1/3} a . \tag{4.128}$$

The Jacobi constant associated with the critical zero-velocity curve that passes through these points can be found from Hill's equations (see Sect. 3.13) and is